

18.03 Class 3, Feb 8, 2010

First order linear equations; systems and signals perspective

- [1] First order linear ODEs
- [2] Bank Accounts; rate and cumulative total
- [3] Systems and signals language
- [4] RC circuits

[1] If I had to name the most important general class of differential equations it would be "linear equations." They will occupy most of this course. Today I'll show you how to model two real world systems by first order linear equations. Both of them involve systems evolving in time. The independent variable is time,  $t$ . If we write  $x$  or  $x(t)$  for the dependent variable, we'll write  $\dot{x}$  for its time-derivative. In these ascii notes I'll continue to write  $x'$  the derivative, though.

Definition: A "linear first order ODE" is one that can be put in the "standard form"

$$r(t) x'(t) + p(t) x(t) = q(t)$$

There is a general analytic method for solving this equation. I'll talk on Wednesday about that. For now note that if  $p$ ,  $q$ , and  $r$  are \*constant\* then the equation is separable --

$$r dx / ( q - px ) = dt$$

Example 0: From recitation: Oryxes with constant growth rate, hunting allowed

$$x' = kx - h \quad h = \text{harvest rate}$$

as in Recitation 1. Notice that in the physical system both  $k$  and  $h$  can vary with time, and that the modeling process is fine with that.

The slide shows an Oryx.

[2] Bank account: I have a bank account. It has  $x$  dollars in it.  $x$  is a function of time. I can deposit money in the account and make withdrawals from it. The bank pays me rent for the money I deposit! This is called interest.

In the old days a bank would pay interest at the end of the month on the balance at the beginning of the month. We can model this mathematically:

With  $\Delta t = 1/12$ , the statement at the end of the month will read:

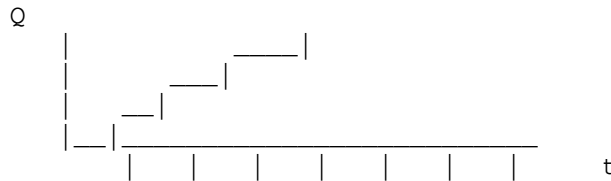
$$x(t + \Delta t) = x(t) + I x(t) \Delta t + [\text{deposits} - \text{withdrawals between } t \text{ and } t + \Delta t]$$

I has units  $(\text{year})^{-1}$  . These days I is typically very small, say  $1\% = 0.01$  . You don't get 1% each month! you get 1/12 of that.

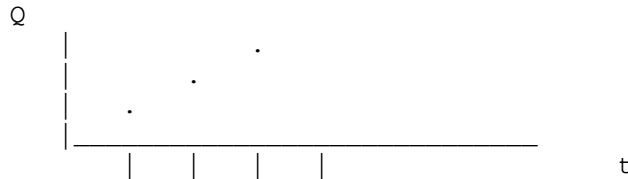
You can think of a withdrawal as a negative deposit, so I will call everything a deposit.

Nowadays interest is usually computed daily. This is a step on the path to the enlightenment afforded by calculus, in which  $\Delta t \rightarrow 0$  .

In order to reach enlightenment, I want to record deposits minus withdrawals as a \*rate\*, in dollars per year. Suppose I contribute \$100 sometime every month, and make no withdrawals. So my total deposits up to time t - my "cumulative total" deposit - has a graph like this



In keeping with letting  $\Delta t \rightarrow 0$  , we should imagine that I am making this contribution continually, at the constant rate of \$1200/year: graph like this

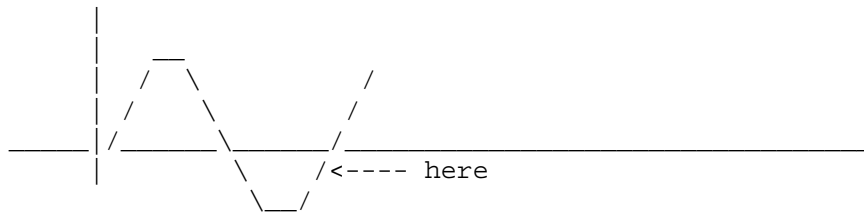


-- a straight line with slope  $1/1200$  :  $Q'(t) = q(t)$  is constant.

In general, say I deposit at the rate of  $q(t)$  dollars per year.  $q(t)$  might vary over time, and might be negative too, from time to time, because withdrawals are merely negative deposits.

Q3.1. At the indicated point of the graph of  $Q(t)$  am I making

- (1) deposits
- (2) withdrawals



Answer: when the slope is positive, not necessarily when  $Q$  is positive: (1).

So (assuming  $q(t)$  is continuous)

$$x(t + \Delta t) \sim x(t) + I x(t) \Delta t + q(t) \Delta t$$

Now subtract  $x(t)$  and divide by  $\Delta t$  :

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} \sim I x + q$$

Now is the moment to let the interest period  $\Delta t$  tend to zero:

$$x' = I x + q$$

Note:  $q(t)$  can certainly vary in time. The interest rate can too. In fact the interest rate might depend upon  $x$  as well: a larger account will probably earn a better interest rate. Neither feature affects the derivation of this equation, but if  $I$  does depend upon  $x$  as well as  $t$ , then the equation we are looking at is no longer linear. So let's say  $I = I(t)$ ,  $q = q(t)$ .

We can put the ODE into standard form:

$$x' - I x = q$$

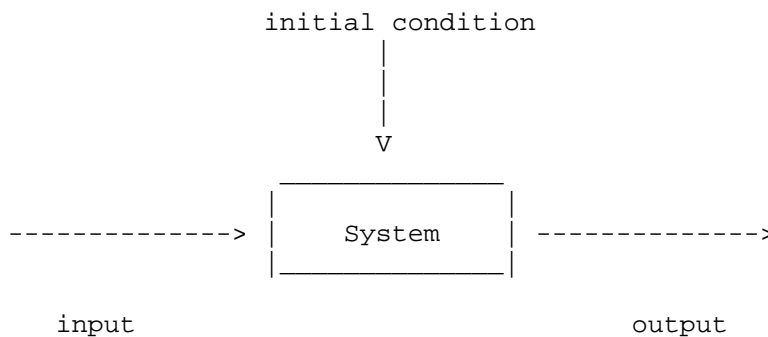
Each symbol represents a function of  $t$ .

### [3] Systems and signals language

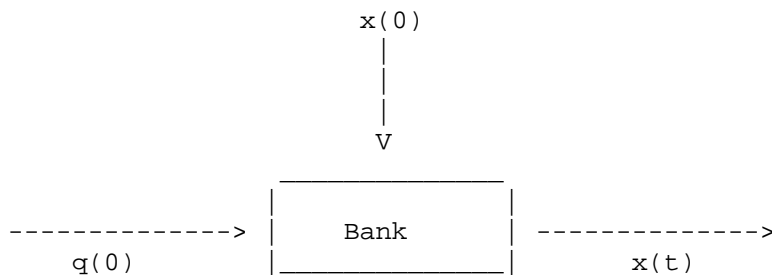
In the standard form, the left hand side represents the SYSTEM: the bank.

The right hand side represents an outside influence on the system: it's a SIGNAL, the "input signal." A "signal" is just a function of time. Here it is my rate of savings.

The system responds to the input signal and yields the function  $x(t)$ , the "output signal" or SYSTEM RESPONSE. Here's a picture:



For example,



We will develop a theory of linear equations, complete with an algorithm for solving them. It's important to recognize them when you see them.

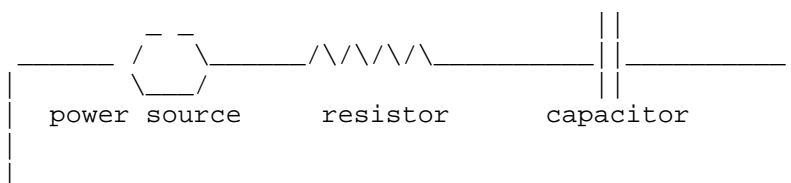
Question 3.2. Which of the following are linear ODE's?

- (a)  $\dot{x} + x^2 = t$
- (b)  $\dot{x} = (t^2 + 1)(x - 1)$
- (c)  $\dot{x} + x = t^2$

- 1. None
- 2. (a) only
- 3. (b) only
- 4. (c) only
- 5. All
- 6. All but (a)
- 7. All but (b)
- 8. All but (c)
- Blank. Don't know.

Answer: (b) and (c) are linear, (a) is not: 6

[4] RC circuits. Suppose we have an electrical circuit like this:



It has a resistor, a capacitor, and a voltage source: it's an RC circuit. This is not a course in electromagnetism or in circuits, but I will use words from that subject. Let's pretend we understand what they mean!

Current flows around the circuit. (Confusingly, if the current flows to the right, the actual electrons flow to the left, because they are negatively charged. It's confusing. Sorry, I didn't invent this.) The current is measured in "amperes" and is denoted by  $I$ . (I don't know what language has a word for current starts with an I!) In this "series" circuit, the current is the

same everywhere but it may vary with time.

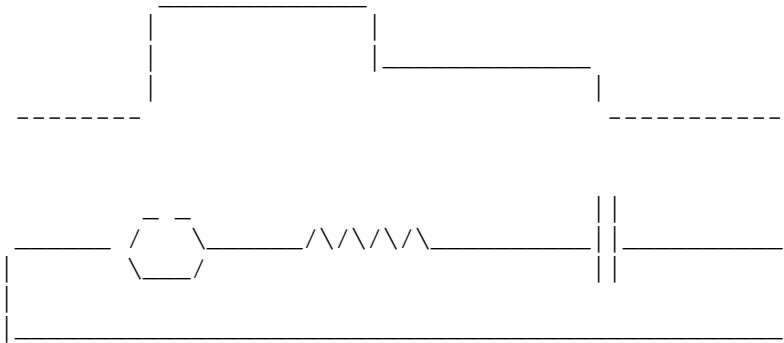
Let's say the positive direction in the circuit is clockwise (ie to the right over the top, for digital clock users). So if current is flowing counterclockwise along the wire, an ammeter would give a negative reading.

The system is powered by a variable power source, which creates a "voltage increase" across it. This what makes current move. Write  $V(t)$  for the voltage INCREASE from the bottom to the top of the source. Write  $V_R$  and  $V_C$  for the voltage DROPS across resistor and capacitor.

"Kirchhoff's voltage law" states that

$$V(t) = V_R(t) + V_C(t)$$

I drew a little graph like this:



to illustrate the KVL.

There is a relationship between the voltage drop across each circuit element and the current flowing through it. The relationship is different for resistors and capacitors:

Resistor:  $V_R(t) = R I(t)$  for a constant  $R$ , the "resistance"

Capacitor:  $V'_C(t) = (1/C) I(t)$  for a constant  $C$ , the "capacitance"

So:

- The voltage drop across the resistor is proportional to the current flowing through it. High resistance means big voltage drop.
- The voltage drop across the capacitor is proportional to the \*integral\* of the current; it results from a buildup of charge on the two plates of the capacitor. High capacitance means lots of space for the charge. A very large capacitor is like no capacitor at all.

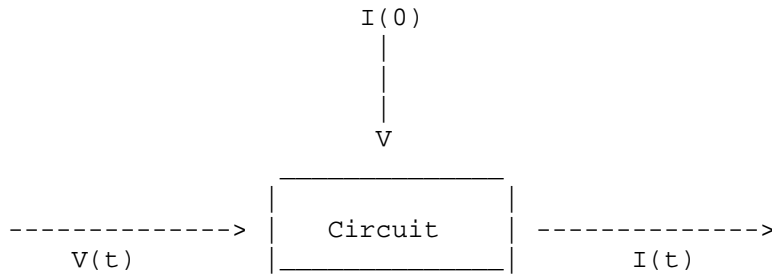
To relate these, differentiate KVL:

$$V'(t) = V'_R(t) + V'_C(t) = R I'(t) + (1/C) I(t)$$

This is a first order linear differential equation for  $I(t)$  :  
In standard form:

$$R I'(t) + (1/C) I(t) = V'(t)$$

The circuit is the system and it is represented by the left hand side.  
The input signal is  $V$  , the voltage increase across the power source.



The \*derivative\* of the input signal is what occurs on the right of the equation.

The system response is the current.

The system is the circuit.

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