General Formula for $\mathbf{n}\,dS$

Suppose S is a surface parametrized by x and y and $\mathbf{N}$ is any vector normal to S (not necessarily unit length). Then $\mathbf{n}\,dS = \dfrac{\mathbf{N}}{\mathbf{N}\cdot\mathbf{k}}\,dx\,dy$. Here $\mathbf{n}$ is the upward unit normal

Example: for the sphere $x^2+y^2+z^2=a^2$ with $\mathbf{N}=\langle x,y,z\rangle$, find $\mathbf{n}\,dS$.

Answer: $\mathbf{n}\,dS = \dfrac{\mathbf{N}}{\mathbf{N}\cdot\mathbf{k}}\,dx\,dy = \langle \dfrac{x}{z}, \dfrac{y}{z}, 1\rangle\,dx\,dy.$

(Just like if we wrote $z=\sqrt{a^2-x^2-y^2}$, $\mathbf{n}\,dS = \langle -z_x, -z_y, 1\rangle\,dx\,dy$.)

18.02SC Multivariable Calculus
Fall 2010