Welcome back to recitation. In this video, what I'd like you to do is use Green's theorem to find the area between one arch of the cycloid that is described by x equals a times the quantity theta minus sine theta, y equals a times the quantity 1 minus cosine theta, and the x-axis. So probably it's helpful for you to draw a picture of this first. So you have one arch of the cycloid defined in terms of theta, the x of theta and y of theta are here, and then the x-axis will be the other bound. And the big thing I want you to notice is you're going to use Green's theorem to find the area. So why don't you pause the video, work on the problem, when you're ready to see my solution you can bring the video back up.

OK, welcome back. So again, what we want to do is we want to use Green's theorem to find an area. OK, and what I'm going to do first is I'm going to try and figure out what the picture looks like and then I'm going to figure out how to use Green's theorem in this region. So lets first draw a sketch of the picture of what's going on.

And because I want the area between one arch of the cycloid and the x-axis, what I really want to do first is figure out what theta values will make y positive, or greater than or equal to 0, and go from there. So if I want y to be greater than or equal to 0, then what I need is 1 minus cosine theta to be greater than or equal to 0. And that's going to happen for every theta from 0 to 2*pi, and at 0 and 2*pi I get 0. So we know we want theta to go between 0 and 2*pi. Those are good parameters to have for theta, because this statement is always true, but I get 0 when theta is 0 and when theta is 2*pi. So I'm going to let theta run between 0 and 2*pi, as I mentioned.

So now let's think about what the picture is. When theta is 0, if I come over to what x is, x is a times 0 minus 0, So x is 0. And y we said was 0. I know that I'll achieve a maximum height at theta equal pi because that's where cosine theta is minus 1. So the 1 minus cosine theta is 2. And so if you look at what the y-value will be, the y-value when theta is pi is 2a. And the value is going to be a times pi minus sine pi. Well, sine of pi is 0. So I just get a*pi. So I should go over a*pi. And I know there's a point at a*pi comma-- I think I said 2a.

And then it's actually going to come back down the same way, so let me draw the first part. This is not a perfect drawing of this. I don't have lots of points. I'm having the smallest amount of points possible to figure out what's happening. So then when theta is 2*pi, 1 minus cosine theta is 0. So I get y equals 0. And then here I get 2 pi minus sine of 2*pi. Sine of 2 pi is 0, so I get 2*pi*a. So I come all the way over here and I get 2*pi*a. Yeah, that's not maybe the most beautiful picture, but it's a good start.

So I'm interested in finding the area of this region. What I want to do then is, to find the area of that region I'm going to use Green's theorem. That's what we asked you to do. So what I want to do is I want to pick a curve that's going to have 2 components. It's going to have C_1, will be this bottom part, and C2 will be this top part. And we need to keep it oriented so when I'm walking along the curve the region is on the left. And then I have to figure out how to use Green's theorem.

So to find the area of this region, all I actually need is the integral over the region of dA. That would give me the area. And so what quantities-- what functions would be good? Sorry. In order to find this. Well, I believe you actually saw in class that you have two options. You have minus y*dx over the closed curve, or you have x*dy over the closed curve. You have one of those two options. Now which one of those is better in this case? We have a choice, so let's make the best choice. Which one of those is better?

Well, if you notice that on these two curves, x is changing on both of them but y is fixed on this one. So it'd be nicer for us if we could just integrate on the curve on this part, which we would be able to do if we were choosing this one. Because y is constant and it is equal to 0 along here. So what you need to do to solve this problem is really, you only need to integrate on C_2, minus y*dx. That's going to give you the entire area.

And so now we have to figure out how to put everything in. Well, C_2-- I'm parameterized in theta-- and C_2 parameterized in theta, if I start here I'm at 2*pi for theta. And if I go this direction I'm at 0 for theta. So what I'm going to do is I'm going to do something that might seem tricky, but it's quite natural. I'm going to replace this minus by a plus and then integrate from 0 to 2*pi. Because I was going to be integrating from 2*pi to 0 and so I just flipped the order and made that sign a plus. So that should be fine.

And then y, I already know what it is. It's a times 1 minus cosine theta. And now I just have to figure out what is dx. So we have to come back over, for one second we're going to come over here and we're going to look at dx. If x is equal to a times theta minus sine theta, then dx is just a times 1 minus cosine theta. So that's nice because that's actually what we already have. So we come back over here-- oh, I should say also with a d theta-- come back over here, dx gives you a squared a, it gives you a squared this, and it gives you a d theta. So that actually gives you everything you need.

Now I am not going to actually write out all the steps in order to do this. But the way you could easily do this, you can pull out the a squared and you're left with a 1 minus cosine theta squared. You might think about squaring all the terms and then you have 1 cosine squared theta to deal with. But ultimately, when you get your answer-- I'm just going to write down what it is-- you get 3*pi a squared. And you'll see you get a little bit of cancellation and it's not too hard to solve at this point. Since it's single variable I'm going to assume I don't need to do it. But you can check your final answer. You should get-- I think, yes. I got 3*pi a squared.

So what we did was we took a problem, again, where we had a curve defined in terms of theta. We had a curve, in x and y, it was parameterized in theta. And we wanted to find an area between that curve and the x-axis. And the reason we could use Green's theorem so easily is because area of a region-- of this connected region here-- is equal to the integral over the closed loop surrounding it of minus y*dx, or alternatively you could have done x*dy.

And because I got to choose, I picked the easier one. And I picked the easier one in this case because y is equal to 0 on one part of the curve. So I just ignored the C_1 because that's where y is 0, and I looked along C_2. I noticed C_2 is parameterized in this direction from theta equals 2*pi to theta equals 0. So that's why I dropped the minus sign and changed it from 0 to 2*pi. That's where that came from.

And then you end up having to just determine what y is and what dx are, in terms of theta. Which we had them already in terms of theta, so we could just explicitly determine everything, do all the substitution and then evaluate the integral. So then, that's all there is to this problem. So hopefully that taught you a little more about Green's theorem. And actually, now you've done problems going from the left to the right and from the right to the left. So you've done a little bit of both. So hopefully this revealed a little bit more to you than you knew before. That's where I'll stop.