18.02 Problem Set 9

At MIT problem sets are referred to as 'psets'. You will see this term used occasionally within the problems sets.

The 18.02 psets are split into two parts 'part I' and 'part II'. The part I are all taken from the supplementary problems. You will find a link to the supplementary problems and solutions on this website. The intention is that these help the student develop some fluency with concepts and techniques. Students have access to the solutions while they do the problems, so they can check their work or get a little help as they do the problems. After you finish the problems go back and redo the ones for which you needed help from the solutions.

The part II problems are more involved. At MIT the students do not have access to the solutions while they work on the problems. They are encouraged to work together, but they have to write their solutions independently.

Part I (10 points)

At MIT the underlined problems must be done and turned in for grading.
The 'Others' are *some* suggested choices for more practice.

A listing like '§1B : $\underline{2}$, 5$\underline{\text{b}}$, $\underline{10}$' means do the indicated problems from supplementary problems section 1B.

1 Green's Theorem
§4D: 1$\underline{\text{c}}$, $\underline{2}$, $\underline{3}$, $\underline{4}$; Others: 1ab, 5

2 Green's Thm. in 'normal form'; flows, divergence & curl.
§4E: 1$\underline{\text{ac}}$, $\underline{2}$, $\underline{4}$, $\underline{5}$; Others: 1b, 3;
§4F: $\underline{4}$; Others: 2, 3

Part II (13 points)

Problem 1 (4: 2,2)

$\mathbf{F}(x,y) \;=\; (y^3 - 6y)\,\mathbf{i} \,+\, (6x - x^3)\,\mathbf{j}$.

a) Using Green's Theorem, find the simple closed curve C for which the integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ (with positive orientation) will have the largest positive value.

b) Compute this largest positive value.

Problem 2 (6: 2,2,2)

In the reading V4.2 (pp.1-3) it is shown that in the context of 2D fluid flows, Green's theorem in normal form combined with the principle of conservation of mass imply that $\mathrm{div}(\mathbf{F})$, the divergence of the flow field $\mathbf{F}(x,y)$, represents the (signed) rate of mass per unit time per unit area which originates at the point (x,y), or the source or sink rate for short. This extends to non-steady flows $\mathbf{F}(x,y,t)$, and leads directly to the *Equation of Continuity* for fluid flows, which is the statement of conservation of mass and hence one of the basic physical principles of fluid dynamics. We'll continue to use ρ for the density (instead of δ used in the Notes).

The divergence of a vector field $\mathbf{F}(x,y,t)$ in this context is defined with respect to the space variables only, that is, if $\mathbf{F}(x,y,t) = \langle M(x,y,t), N(x,y,t)\rangle$ is a smooth vector field, then $\mathrm{div}(\mathbf{F}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$.

Then for the case of a flow field $\mathbf{F}(x,y,t) = \rho(x,y,t)\,\mathbf{v}(x,y,t)$ with density $\rho(x,y,t)$ and velocity $\mathbf{v}(x,y,t)$, the equation of continuity reads

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\mathbf{F}) \;=\; 0.$$

Note that for *steady* flows, which by definition means $\rho = \rho(x,y)$ and $\mathbf{v} = \mathbf{v}(x,y)$, the equation of continuity holds if and only if $\mathrm{div}(\mathbf{F}) = 0$. Thus conservation of mass for steady flows is equivalent to the absence of any sources or sinks, which makes sense.

a) For non-steady flows, assuming that the physical interpretation of $\mathrm{div}(\mathbf{F})$ is the same as in the case of steady flows (at each time t), explain why the equation of continuity is in fact the statement of conservation of mass.

Hint: take an arbitrary bounded region $\mathcal{R}$ and integrate both terms of the continuity equation over $\mathcal{R}$. Then use Green's theorem in normal form.

b) Let $g(x,y,t)$ be a smooth scalar function, and again define the gradient of $g(x,y,t)$ in this case to be with respect to just the space variables: $\nabla g = \langle g_x,\, g_y\rangle$. Then if and $\mathbf{G}(x,y,t) = \langle M(x,y,t), N(x,y,t)\rangle$ is a smooth vector field, use the product rule to show that

$$\mathrm{div}\,(\mathrm{g}\;\mathbf{G}) \;=\; \mathrm{g}\;\mathrm{div}(\mathbf{G}) \;+\; \mathbf{G}\cdot\nabla \mathrm{g}$$

c) Refer to the definition of the convective derivative $\dfrac{Df}{Dt}$ given in p-set 5 #2, and the definition of incompressibility for flows $\dfrac{D\rho}{Dt} \;=\; 0$, as given in p-set 5 #3.

Combining: the equation of continuity; the result of part(b) above; and the result of p-set 5 #2, show that the flow $\mathbf{F}(x, y, t) = \rho(x, y, t)\, \mathbf{v}(x, y, t)$ is incompressible if and only if

$$\mathrm{div}(\mathbf{v}) \;=\; 0.$$

This is thus an equivalent condition for the incompressibility of a flow.

Problem 3 (3)

Sketch each of the following non-steady flows.
Verify that it satisfies the equation of continuity.
(*Suggestion*: Use the expanded form of the equation of continuity found in problem 2(c) above.)
Then test it to determine whether it is incompressible, and if so, whether it is also stratified (see p-set 5 #3(b)):

(i) $\mathbf{v}(x, y, t) = t\,\langle -y, x\rangle, \quad \rho(x, y, t) \;=\; \sqrt{x^2 + y^2}$

(ii) $\mathbf{v}(x, y, t) = \frac{1}{1+t}\,\langle x, -y\rangle, \quad \rho(x, y, t) = xy$

(iii) $\mathbf{v}(x, y, t) = t\,\langle x, y\rangle, \quad \rho(x, y, t) = \mathrm{e}^{-t^2}.$