Identifying Potential Functions

1. Show $\mathbf{F} = \langle 3x^2 + 6xy, 3x^2 + 6y \rangle$ is conservative and find the potential function f such that $\mathbf{F} = \boldsymbol{\nabla} f$.

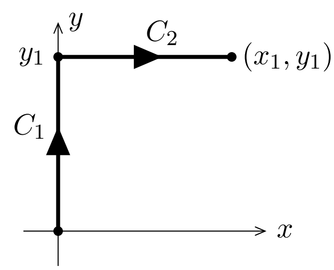
Answer: First, $M_y = 6x = N_x$. Since $\mathbf{F}$ is defined for all (x, y), $\mathbf{F}$ is conservative.

Method 1 (for finding f): Use

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(P_1) - f(P_0) \;\Rightarrow\; f(P_1) = f(P_0) + \int_C \mathbf{F} \cdot d\mathbf{r}.$$

$P_1 = (x_1, y_1)$ must be arbitrary. We can fix P_0 and C any way we want.

For this problem take $P_0 = (0, 0)$ and C as the path shown.



Path for problem 1

$C_1 : x = 0, y = y, \;\Rightarrow\; dx = 0, dy = dy$

$C_2 : x = x, y = y_1, \;\Rightarrow\; dx = dx, dy = 0$

$$\begin{aligned} \Rightarrow \quad f(x_1, y_1, z_1) - f(0, 0, 0) &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy \\ &= \int_0^{x_1} M(x, 0)\,dx + \int_0^{y_1} N(x_1, y)\,dy \\ &= \int_0^{y_1} 6y\,dy + \int_0^{x_1} 3x^2 + 6y_1\,dx = 3y_1^2 + x_1^3 + 3x_1^2 y_1 \end{aligned}$$

$\Rightarrow\; f(x_1, y_1) - f(0, 0) = 3y_1^2 + x_1^3 + 3x_1^2 y_1 = 3y_1^2 + x_1^3 + 3x_1^2 y_1.$

$\Rightarrow\; \boxed{f(x, y) = 3y^2 + x^3 + 3x^2 y + C.}$

Method 2: $f_x = 3x^2 + 6xy \;\Rightarrow\; f = x^3 + 3x^2 y + g(y).$

$\Rightarrow\; f_y = 3x^2 + g'(y) = 3x^2 + 6y \;\Rightarrow\; g'(y) = 6y \;\Rightarrow\; g(y) = 3y^2 + C.$

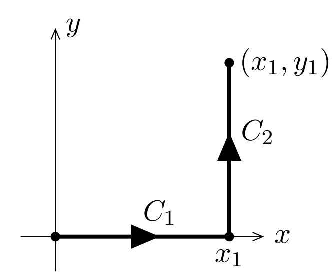
$\Rightarrow\; \boxed{f(x, y) = x^3 + 3x^2 y + 3y^2 + C.}$

In general method 1 is preferred because in 3 dimensions it will be easier.

2. Let $\mathbf{F} = (x + xy^2)\,\mathbf{i} + (x^2 y + 3y^2)\,\mathbf{j}$. Show $\mathbf{F}$ is a gradient field and find the potential function using both methods.

Answer: We have $M(x, y) = x + xy^2$ and $N(x, y) = x^2 y + 3y^2$, so $M_y = 2xy = N_x$ and $\mathbf{F}$ is defined on all (x, y). Thusy, by Theorem 1, $\mathbf{F}$ is conservative.

Method 1: Use the path shown.



Path for problem 2

$f(P_1) - f(0, 0) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy.$

$C_1 : x = x,\; y = 0, \;\Rightarrow\; dx = dx,\; dy = 0 \;\Rightarrow\; M(x, 0) = x.$

$C_2 : x = x_1,\; y = y, \;\Rightarrow\; dx = 0,\; dy = dy \;\Rightarrow\; N(x_1, y) = x_1^2 y + 3y^2.$

$$\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} \quad = \quad \int_0^{x_1} x\,dx + \int_0^{y_1} x_1^2 y + 3y^2\,dy \quad = \quad x_1^2/2 + x_1^2 y_1^2/2 + y_1^3.$$

$\Rightarrow\; f(x_1, y_1) - f(0, 0) = x_1^2/2 + x_1^2 y_1^2/2 + y_1^3$

$\Leftrightarrow\; \boxed{f(x, y) = x^2/2 + x^2 y^2/2 + y^3 + C.}$

Method 2: $f_x = x + y^2 \;\Rightarrow\; f = x^2/2 + x^2 y^2/2 + g(y)$

$\Rightarrow\ f_y = x^2y + g'(y) = x^2y + 3y^2\ \Rightarrow\ g'(y) = 3y^2\ \Rightarrow\ g(y) = y^3 + C.$

$\Rightarrow\ \boxed{f(x,y) = x^2/2 + x^2y^2/2 + y^3 + C.}$