Line Integrals of Vector Fields

In lecture, Professor Auroux discussed the non-conservative vector field

$$\mathbf{F} = \langle -y, x \rangle.$$

For this field:

1. Compute the line integral along the path that goes from $(0, 0)$ to $(1, 1)$ by first going along the x-axis to $(1, 0)$ and then going up one unit to $(1, 1)$.

__**Answer:**__ To compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ we break the curve into two pieces, then add the line integrals along each piece.

First, fix $y = 0$ (so $dy = 0$) and let x range from 0 to 1.

$$\int_{x=0}^{x=1} \mathbf{F} \cdot d\mathbf{r} = \int_{x=0}^{x=1} -y\, dx + x\, dy = \int_0^1 0\, dx = 0.$$

Next, fix $x = 1$ (so $dx = 0$) and let y range from 0 to 1:

$$\int_{y=0}^{y=1} \mathbf{F} \cdot d\mathbf{r} = \int_{y=0}^{y=1} -y\, dx + 1\, dy = 1.$$

We conclude that $\int_C \mathbf{F} \cdot d\mathbf{r} = 1$.

2. Compute the line integral along the path from $(0, 0)$ to $(1, 1)$ that first goes up the y-axis to $(0, 1)$.

__**Answer:**__ Again we split the curve into two parts. We start by fixing $x = 0$ (so $dx = 0$) and letting y range from 0 to 1:

$$\int_{y=0}^{y=1} -y\, dx + 0\, dy = 0.$$

Next fix $y = 1$ and let x range from 0 to 1:

$$\int_{x=0}^{x=1} -1\, dx + x\, dy = -x\Big|_0^1 = -1.$$

Here, $\int_C \mathbf{F} \cdot d\mathbf{r} = -1$.

3. Should you expect your answers to the preceding problems to be the same? Why or why not?

__**Answer:**__ If $\mathbf{F}$ were conservative, the value of a line integral starting at $(0, 0)$ and ending at $(1, 1)$ would be independent of the path taken. We know from lecture that $\mathbf{F}$ is non-conservative, so we don't expect line integrals along different paths to have the same values.

4. Compute the line integral of $\mathbf{F}$ along a path that runs counterclockwise around the unit circle.

Answer: We parametrize C by $x = \cos\theta$, $y = \sin\theta$ with $0 \le \theta < 2\pi$. Then $dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$, and:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\theta=0}^{\theta=2\pi} -\sin\theta\, dx + \cos\theta\, dy = \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta = 2\pi.$$

5. Should your answer to the previous problem be 0? Why or why not?

Answer: The vector field is not conservative, so its line integral around a closed curve need not be zero.

Answer the following questions for the field

$$\mathbf{F} = \langle 0, x \rangle.$$

6. Compute the line integral along the path that goes from $(0,0)$ to $(1,1)$ by first going along the x-axis to $(1,0)$ and then going up one unit to $(1,1)$.

Answer: We split the curve into two pieces as in problem (1).

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left(\int_{x=0}^{x=1} 0\, dx + x\, dy \right) + \left(\int_{y=0}^{y=1} 0\, dx + 1\, dy \right) = 1.$$

7. Compute the line integral along the path from $(0,0)$ to $(1,1)$ which first goes up the y-axis to $(0,1)$.

Answer: Proceed as in problem (2):

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left(\int_{y=0}^{y=1} 0\, dx + 0\, dy \right) + \left(\int_{x=0}^{x=1} 0\, dx + x\, dy \right) = 0.$$

8. Compute the line integral of $\mathbf{F}$ along the line segment from $(0,0)$ to $(1,1)$.

Answer: Parametrize C by $x = y = t$ where $0 \le t \le 1$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{t=1} 0\, dt + t\, dt = \left.\frac{t^2}{2}\right|_0^1 = 1/2.$$

9. Is the vector field $\mathbf{F} = \langle 0, x \rangle$ conservative? How do you know?

Answer: The field $\mathbf{F}$ is not conservative. If it were, the line integrals in problems 6, 7 and 8 would depend only on the endpoints of C and so would have the same values.