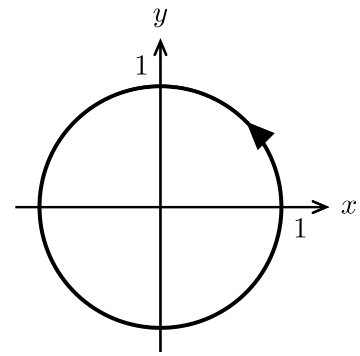
Geometric Approach to Line Integrals

Line integrals are intrinsically geometric, so we should sometimes be able to use geometric reasoning to avoid the tedious calculations used in computing certain line integrals. The geometry can also give us some insight into the situation that calculation sometimes obscures.



We start with a line integral that we compute directly,

Example 1: Evaluate $I = \oint_C -y\,dx + x\,dy$ where C is the unit circle traversed in a counterclockwise (CCW) direction.
Parametrization: $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$.
$\Rightarrow\; dx = -\sin t\,dt,\; dy = \cos t\,dt.$
$\Rightarrow\; I = \int_0^{2\pi} -\sin t(-\sin t)\,dt + \cos t(\cos t)\,dt = \int_0^{2\pi} dt = 2\pi.$

The intrinsic formula Recall that we know $\dfrac{d\mathbf{r}}{dt} = \mathbf{T}\dfrac{ds}{dt}$, where $\mathbf{T}$ = unit tangent and s = arclength. Removing the dt gives $d\mathbf{r} = \mathbf{T}\,ds$. We can use this in our formula for line integrals and get a form that we call the *intrinsic formula*

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

Example 2: Redo example 1 using the intrinsic formula: $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$.
$\mathbf{T}$ = unit tangent $\Rightarrow$ $\mathbf{T} = -y\,\mathbf{i} + x\,\mathbf{j}$ (on the unit circle $x^2 + y^2 = 1$).
$\Rightarrow\; \mathbf{F}\cdot\mathbf{T} = y^2 + x^2 = 1$ (on the unit circle) $\Rightarrow$ $I = \int_C ds$ = arclength of circle $= 2\pi$.
Lesson: it can pay to think geometrically.