18.02 Problem Set 7

At MIT problem sets are referred to as 'psets'. You will see this term used occasionally within the problems sets.

The 18.02 psets are split into two parts 'part I' and 'part II'. The part I are all taken from the supplementary problems. You will find a link to the supplementary problems and solutions on this website. The intention is that these help the student develop some fluency with concepts and techniques. Students have access to the solutions while they do the problems, so they can check their work or get a little help as they do the problems. After you finish the problems go back and redo the ones for which you needed help from the solutions.

The part II problems are more involved. At MIT the students do not have access to the solutions while they work on the problems. They are encouraged to work together, but they have to write their solutions independently.

Part I (15 points)

At MIT the underlined problems must be done and turned in for grading.
The 'Others' are *some* suggested choices for more practice.

A listing like '§1B : $\underline{2}$, 5$\underline{b}$, $\underline{10}$' means do the indicated problems from supplementary problems section 1B.

1 Double Integrals.
§3A: 2$\underline{c}$, 3$\underline{b}$, 4$\underline{c}$, 5$\underline{a}$; Others: 1, 2ab, 3a, 4b, 5bc, 6, 7

2 Double Integrals in polar coordinates. Applications.
§3B: 1$\underline{a}$, 2$\underline{d}$, 3$\underline{ac}$; Others: 1cd, 2b, 3b
§3C: $\underline{1}$, 2$\underline{a}$, $\underline{4}$; Others: 2b, 3

3 Change of variables in double integrals.
§3D: $\underline{1}$, $\underline{2}$, $\underline{4}$; Others: 3

Part II (25 points)

Problem 1 (3: 1,2)
a) Sketch the solid in the first octant bounded by the xy, yz and xz coordinate planes, the plane $x + y = 4$ and the surface $z = \sqrt{4 - x}$.
b) Find the volume of the solid of part (a).

Problem 2 (5: 3,2)
The equation of a surface of revolution obtained by spinning the (1-D) graph of the function $z = f(y)$ in the y-z plane around the z-axis is given in polar coordinates by $z = f(r, \theta) = f(r)$ (that is, there is no dependence on θ). Assume that $h := f(0) > 0$.
a) Show that the formula which you get by using the double integral in polar coordinates for the volume under the graph of this surface of revolution and over the x-y plane is the same as that given by the "shell method" in single-variable calculus.
b) Illustrate with a sketch.

Problem 3 (4: 2,1,1)
a) Find the formula for the centroid (center of mass assuming density = 1) of a uniform plane region in the shape of circular sector of radius a and central angle θ (in terms of a and θ).
b) Write down the formula for the centroid of a uniform plane region in the shape of a isoceles triangle with height a and angle θ between the two equal sides. (You may use the formula from elementary geometry or compute it out using the integral.)
c) If you align the two regions above using the same values for a and θ and with the central angle of the sector coinciding with the apex angle of the triangle, which centroid will be closer to the center (= the apex)? What do the math and the physics predict and why, and are they consistent?

Background for problems 4 and 5:
In fluid mechanics the *fluid flow map* φ is defined as follows: if (x, y, z) is the position of a point mass in the flow at time $t = 0$, then $(X, Y, Z) = \varphi(x, y, z, t)$ is the downstream position of that same point mass after an elapsed time t.
The standard assumptions on φ are that it is smooth and one-to-one.
We will call a flow *volume incompressible* if for any bounded space region $\mathcal{R}$ in the flow, the volume of $\varphi(\mathcal{R}, t)$ is the same as the volume of $\mathcal{R}$ for all t. In other words, if $\mathcal{R}_t = \varphi(\mathcal{R}, t) = \{\varphi(x, y, z, t) \mid (x, y, z) \text{ in } \mathcal{R}\}$ is the the region formed by the points from $\mathcal{R}$ which have been carried downstream by the flow, then $\mathcal{R}_t$ can have a different shape but must have the same volume at all times, if the flow is 'v-i'.
In problems 4 & 5, we'll take the simpler case of a 2D flow (which could be e.g. a 2D section of a flow in 3D). Let $(X(x, y, t), Y(x, y, t)) = \varphi(x, y, t)$. Note that then by definition, the velocity vectors $\mathbf{v}(x, y, t)$ of the flow are given by $\mathbf{v}(x, y, t) = \langle \frac{\partial X}{\partial t}, \frac{\partial Y}{\partial t} \rangle$.
A v-i flow in this case is one that preserves area, since area is the 2D version of volume.

For a fixed value of t, let $J(x,y,t) = \dfrac{\partial(X,Y)}{\partial(x,y)}$ be the Jacobian of the transformation $(x,y) \mapsto (X(x,y,t), Y(x,y,t))$. The general change-of-variables formula says that if a region $\mathcal{R}$ goes to a region $\mathcal{R}'$ by a transformation $(x,y) \mapsto (X,Y)$ with Jacobian $\frac{\partial(X,Y)}{\partial(x,y)}$, then the areas of $\mathcal{R}$ and $\mathcal{R}'$ are related by $A(\mathcal{R}') = \iint_{\mathcal{R}} \mid J(x,y) \mid \, dA$. Here this gives that $A(\mathcal{R}_t) = \iint_{\mathcal{R}} \mid J(x,y,t) \mid \, dA$, and therefore that a 2D flow is v-i if and only if $\mid J(x,y,t) \mid = 1$ for all (x,y,t).

In problems 5 & 6 we will look at three examples of 2D flows, v-i and non-vi, in order to illustrate this idea.

Example A: $\varphi(x,y,t) = ((1+t)x, (1+t)y)$;
$\mathcal{R} =$ the triangle with vertices at $(0,0)$, $(1,1)$ and $(1,-1)$.

Example B: $\varphi(x,y,t) = (x\cos t - y\sin t, x\sin t + y\cos t)$;
$\mathcal{R} =$ the triangle with vertices at $(0,0)$, $(2,0)$ and $(2,1)$.

Example C: $\varphi(x,y,t) = \big((1+t)x, (\frac{1}{1+t})y\big)$;
$\mathcal{R} =$ the rectangle with vertices at $(1,1)$, $(1,4)$, $(2,1)$ and $(2,4)$.

Problem 4 (4: 2,2)
In each of the cases A, B and C:
i) compute the Jacobian $J(x,y,t)$
ii) compute the area $A(\mathcal{R}_t)$

Problem 5 (9: 3,3,3)
In each of the cases A, B and C:
i) Sketch the pattern of the flow paths over time, including some starting from points in $\mathcal{R}$.
ii) Compute the velocity vectors of the flow and sketch in a few on the flow lines.
iii) Sketch the regions $\mathcal{R}$ and $\mathcal{R}_t$ and check this against the areas calculated in problem 5 to see if it looks correct in each case. Use the following values for t:
A: $t = 2$, B: $t = \frac{\pi}{2}$, and C: $t = 3$.
Suggestion for sketching $\mathcal{R}_t$: see where the corners of $\mathcal{R}$ end up on $\mathcal{R}_t$.
For the flow lines in case C, also note that $X(x,y,t)Y(x,y,t) = xy$ for all values of t.
iv) Identify which flows are v-i, using the computed results (and the sketches).
v) In the cases A and B, describe the what the flow is doing.

Note: after we learn Green's Theorem in normal form in 2D, or equivalently the Divergence Theorem in 3D, we'll be able to prove that incompressibility and (our ad-hoc terminology) 'volume-incompressibility' are in fact equivalent conditions.