EXAM 2 PRACTICE MATERIALS

Definitions and Theorems

(1) Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ be a vector field. State what it means for $\mathbf{f}$ to be differentiable.

(2) Assume $\mathbf{f}$ as above is differentiable. Give the definition for $D_k\mathbf{f}$.

(3) Let $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m$ be defined such that $\mathbf{h}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$ where $\mathbf{g} : \mathbb{R}^k \to \mathbb{R}^m, \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^k$.
State the chain rule in this generality.
Then use more appropriate notation to describe the specific case when $n = m = 1$ and $k \neq 1$.
Do the same for when $n = k = 1$ and $m \neq 1$.

(4) State the implicit function theorem for scalar fields.

(5) State the second derivative test for $f : \mathbb{R}^2 \to \mathbb{R}$.

(6) State Taylor's Theorem for $f : \mathbb{R}^2 \to \mathbb{R}$.

(7) State the two fundamental theorems of calculus for line integrals.

(8) State the necessary and sufficient condition for a vector field to be a gradient vector field on an open, convex $S \subset \mathbb{R}^n$. Now state a necessary and sufficient condition for a vector field to be a gradient field when S is open and connected.

(9) Define a bounded set of content zero.

(10) State the definition of an integrable function on a rectangle in $\mathbb{R}^2$.

(11) State Fubini's Theorem.

Problems

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a scalar field. For each of the following questions, answer "yes" or "no". If the answer is "yes", cite a theorem or give a brief sketch of a proof. If the answer is "no", provide a counterexample.
  (a) Suppose $f'(\mathbf{a};\mathbf{x})$ exists for all $\mathbf{x} \in \mathbb{R}^2$. Is f continuous at $\mathbf{a}$?
  (b) Suppose $D_1 f, D_2 f$ both exist at $\mathbf{a}$. Does $f'(\mathbf{a};\mathbf{x})$ exist for all $\mathbf{x} \in \mathbb{R}^2$?
  (c) Suppose f is differentiable at $\mathbf{a}$. Is f continuous at $\mathbf{a}$?
  (d) Suppose $D_1 f, D_2 f$ both exist at $\mathbf{a}$ and are continuous in a neighborhood of $\mathbf{a}$. Is f continuous at $\mathbf{a}$?

(2) Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathbf{f}(x, y) = (x^2 + y, 2x + y^2)$.
Find $D\mathbf{f}$ and determine the values of (x, y) for which $\mathbf{f}$ is NOT invertible.
Given that $\mathbf{f}$ is invertible at $(0, 0)$, let $\mathbf{g}$ be its inverse. Find $D\mathbf{g}(0, 0)$.

(3) Let $f(x, y, z) = 2x^2y + xy^2z + xyz$ and consider the level surface $f(x, y, z) = 4$.
Find the tangent plane at $(x, y, z) = (1, 1, 1)$.
Explain why it is possible to find a function $g(x, y)$, defined in a neighborhood of $(x, y) = (1, 1)$ such that a neighborhood of $(1, 1, 1)$ on the surface $f(x, y, z) = 4$ can be written as a graph $(x, y, g(x, y))$.

(4) Find all extreme values for $f(x, y, z) = x^2 + 2y^2 + 4z^2$ subject to the constraint $x + y + z = 7$. Justify whether the extreme values are maximum or a minimum.

(5) Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field with $\mathbf{f} = (f_1, f_2, \ldots, f_n)$. We define the divergence of $\mathbf{f}$ such that

$$div(\mathbf{f}) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}.$$

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a smooth scalar field. Prove that

$$div(\nabla g) = \sum_{i=1}^{n} \frac{\partial^2 g}{\partial x_i^2}.$$

(6) Assume f, g are integrable on the rectangle $Q \subset \mathbb{R}^2$ and let $a, b \in \mathbb{R}$. Given the linearity of the integral for step functions, prove $\int\int_Q (af + bg)dxdy = a\int\int_Q f dxdy + b\int\int_Q g dxdy$.