Derivatives of vector functions.

Recall that if $\underline{x}$ is a point of R^n and if $f(\underline{x})$ is a scalar function of $\underline{x}$, then the derivative of f (if it exists) is the vector

$$\vec{\nabla} f = (D_1 f, \ldots, D_n f)$$

$$= (\partial f/\partial x_1, \ldots, \partial f/\partial x_n).$$

For some purposes, it will be convenient to denote the derivative of f by a row matrix rather than by a vector. When we do this, we usually denote the derivative by Df rather than $\vec{\nabla} f$. Thus

$$Df(\underline{a}) = [D_1 f(\underline{a}) \quad D_2 f(\underline{a}) \quad \ldots \quad D_n f(\underline{a})].$$

If we use this notation, the definition of the derivative takes the following form:

$$f(\underline{a}+\underline{h}) - f(\underline{a}) = Df(\underline{a}) \cdot \underline{h} + \varepsilon(\underline{h}) \|\underline{h}\|,$$

where $\varepsilon(\underline{h}) \longrightarrow 0$ as $\underline{h} \longrightarrow \underline{0}$. Here the dot denotes matrix multiplication, so we must write $\underline{h}$ as a column matrix in order for the formula to work;

$$\underline{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}.$$

This is the formula that will generalize to vector functions $\underline{f}$.

Definition. Let S be a subset of R^n. If $\underline{f} : S \longrightarrow R^k$, then $\underline{f}(\underline{x})$ is called a vector function of a vector variable. In scalar form, we can write $\underline{f}(\underline{x})$ out in the form

$$\underline{f}(\underline{x}) = (f_1(x_1,\ldots,x_n),\ldots,f_k(x_1,\ldots,x_n)).$$

Said differently, $\underline{f}$ consists of "k real-valued functions of n variables." Suppose now that $\underline{f}$ is defined in an open ball about the point $\underline{a}$. We say that $\underline{f}$ is differentiable at $\underline{a}$ if each of the functions $f_1(\underline{x}),\ldots,f_k(\underline{x})$ is differentiable at $\underline{a}$ (in the sense already defined). Furthermore, we define the derivative of $\underline{f}$ at $\underline{a}$ to be the matrix

$$D\underline{f}(\underline{a}) = \begin{bmatrix} D_1f_1(\underline{a}) & D_2f_1(\underline{a}) & \cdots & D_nf_1(\underline{a}) \\ D_1f_2(\underline{a}) & D_2f_2(\underline{a}) & \cdots & D_nf_2(\underline{a}) \\ & & \cdots & \\ D_1f_k(\underline{a}) & D_2f_k(\underline{a}) & \cdots & D_nf_k(\underline{a}) \end{bmatrix}.$$

That is, $D\underline{f}(\underline{a})$ is the matrix whose $i^{\underline{th}}$ row is the derivative $Df_i(\underline{a})$ of the $i^{\underline{th}}$ coordinate function of $\underline{f}$.

Said differently, the derivative $D\underline{f}(\underline{a})$ of $\underline{f}$ at $\underline{a}$ is the k by n matrix whose entry in row i and column j is

$$D_jf_i(\underline{x}) = \partial f_i/\partial x_j;$$

it is often called the Jacobian matrix of $\underline{f}(\underline{x})$. Another notation for this matrix is the notation

$$\frac{\partial(f_1,\ldots,f_k)}{\partial(x_1,\ldots,x_n)} .$$

With this notation, many of the formulas we proved for a scalar function $f(\underline{x})$ hold without change for a vector function $\underline{f}(\underline{x})$. We consider some of them here:

Theorem 1. *The function $\underline{f}(\underline{x})$ is differentiable at $\underline{a}$ if and only if*

$$\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a}) = D\underline{f}(\underline{a}) \cdot \underline{h} + \underline{E}(\underline{h})\|\underline{h}\|,$$

where $\underline{E}(\underline{h}) \longrightarrow 0$ as $\underline{h} \longrightarrow \underline{0}$.
(Here $\underline{f}$, $\underline{h}$, and $\underline{E}$ are written as column matrices.)

Proof: Both sides of this equation represent column matrices. If we consider the $i^{\underline{th}}$ entries of these matrices, we have the following equation:

$$f_i(\underline{a}+\underline{h}) - f_i(\underline{a}) = Df_i(\underline{a}) \cdot \underline{h} + E_i(\underline{h})\|\underline{h}\| .$$

Now $\underline{f}$ is differentiable at $\underline{a}$ if and only if each function f_i is. And f_i is differentiable at $\underline{a}$ if and only if $E_i(\underline{h}) \longrightarrow 0$ as $\underline{h} \longrightarrow \underline{0}$. But $E_i(\underline{h}) \longrightarrow 0$ as $\underline{h} \longrightarrow \underline{0}$, for each i, if and only if $\underline{E}(\underline{h}) \longrightarrow \underline{0}$ as $\underline{h} \longrightarrow \underline{0}$. $\square$

Theorem 2. *If $\underline{f}(\underline{x})$ is differentiable at $\underline{a}$, then $\underline{f}$ is continuous at* a.

Proof. If $\underline{f}$ is differentiable at $\underline{a}$, then so is each function f_i. Then in particular, f_i is continuous at $\underline{a}$, whence $\underline{f}$ is continuous at $\underline{a}$.

The general chain rule.

Before considering the general chain rule, let us take the chain rule we have already proved and reformulate it in terms of matrices.

Assume that $f(\underline{x}) = f(x_1,\ldots,x_n)$ is a scalar function defined in an open ball about $\underline{a}$, and that $\underline{x}(t) = (x_1(t),\ldots, x_n(t))$ is a parametrized curve passing through $\underline{a}$. Let $\underline{x}(t_0) = \underline{a}$. If $f(\underline{x})$ is differentiable at $\underline{a}$, and if $\underline{x}(t)$ is differentiable at t_0, and we have shown that the composite $f(\underline{x}(t))$ is differentiable at t_0, and its derivative is given by the equation

$$\frac{d}{dt}\, f(\underline{x}(t)) = \vec{\nabla} f(\underline{x}(t)) \cdot \underline{x}'(t)$$

when $t = t_0$.

We can rewrite this formula in scalar form as follows:

$$\frac{d}{dt}\, f(\underline{x}(t)) = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt} \;;$$

or we can rewrite it in the following matrix form:

$$\frac{d}{dt}\ f(\underline{x}(t)) = \begin{bmatrix}\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n}\end{bmatrix} \cdot \begin{bmatrix} dx_1/dt \\ \vdots \\ dx_n/dt \end{bmatrix}$$

Recalling the definition of the Jacobian matrix $D\underline{f}$, we see that the latter formula can be written in the form

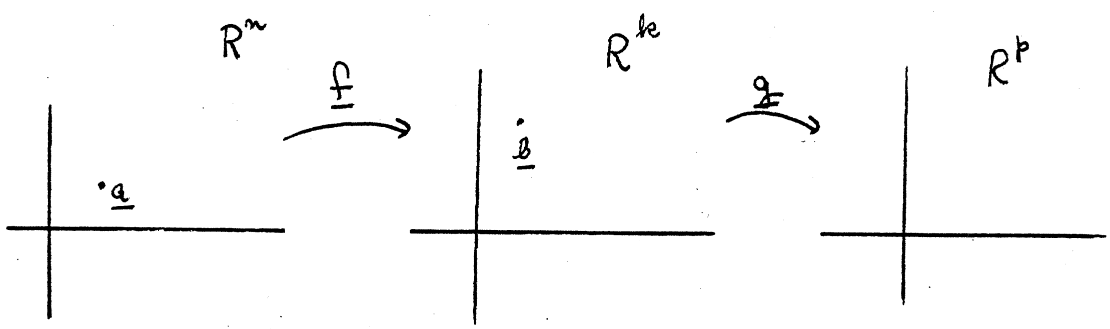
$$\frac{d}{dt}\ f(\underline{x}(t)) = Df(\underline{x}(t)) \cdot D\underline{x}(t).$$

(Note that the matrix Df is a row matrix, while the matrix $D\underline{x}$ is by its definition a column matrix.)

This is the form of the chain rule that we find especially useful, for it is the formula that generalizes to higher dimensions.

Let us now consider a composite of vector functions of vector variables. For the remainder of this section, we assume the following:

<u>Suppose $\underline{f}$ is defined on an open ball in R^n about $\underline{a}$, taking values in R^k, with $\underline{f}(\underline{a}) = \underline{b}$. Suppose $\underline{g}$ is defined in an open ball about $\underline{b}$, taking values in R^p. Let $\underline{F}(\underline{x}) = \underline{g}(\underline{f}(\underline{x}))$ denote the composite function.</u>



We shall write these functions as

$$\underline{f}(\underline{x}) = \underline{f}(x_1,\ldots,x_n) \quad \text{and} \quad \underline{g}(\underline{y}) = \underline{g}(y_1,\ldots,y_k).$$

If $\underline{f}$ and $\underline{g}$ are differentiable at $\underline{a}$ and $\underline{b}$ respectively, it is easy to see that the partial derivatives of $\underline{F}(\underline{x})$ exist at $\underline{a}$, and to calculate them. After all, the $i^{\underline{th}}$ coordinate function of $\underline{F}(\underline{x})$ is given by the equation

$$F_i(\underline{x}) = g_i(\underline{f}(\underline{x})).$$

If we set each of the variables x_ℓ, except for the single variable x_j, equal to the constant a_ℓ, then both sides are functions of x_j alone. The chain rule already proved then gives us the formula

$$(*) \qquad \frac{\partial F_i}{\partial x_j} = \frac{\partial g_i}{\partial y_1}\frac{\partial f_1}{\partial x_j} + \frac{\partial g_i}{\partial y_2}\frac{\partial f_2}{\partial y_2} + \cdots + \frac{\partial g_i}{\partial y_k}\frac{\partial f_k}{\partial x_j}.$$

Thus

$$D_jF_i = [D_1g_i \quad D_2g_i \quad \cdots \quad D_kg_i] \cdot \begin{bmatrix} D_jf_1 \\ D_jf_2 \\ \vdots \\ D_jf_k \end{bmatrix}.$$

$$= [i^{\underline{th}} \text{ row of } D\underline{g}] \cdot \begin{bmatrix} j^{\underline{th}} \text{ column} \\ \text{of } D\underline{f} \end{bmatrix}.$$

domains, then the composite $\underline{F}(\underline{x}) = \underline{g}(\underline{f}(\underline{x}))$ is continuously differentiable on its domain, and

$$D\underline{F}(\underline{x}) = D\underline{g}(\underline{f}(\underline{x})) \cdot Df(\underline{x}).$$

This theorem is adequate for all the chain-rule applications we shall make.

Note: The matrix form of the chain rule is nice and neat, and it is useful for theoretical purposes. In practical situations, one usually uses the scalar formula (*) when one calculates partial derivatives of a composite function, however.

The following proof is included solely for completeness ; we shall not need to use it:

Theorem 4. Let $\underline{f}$ and $\underline{g}$ be as above. If f is differentiable at $\underline{a}$ and $\underline{g}$ is differentiable at $\underline{b}$, then

$$\underline{F}(\underline{x}) = \underline{g}(\underline{f}(\underline{x}))$$

is differentiable at $\underline{a}$, and

$$D\underline{F}(\underline{a}) = D\underline{g}(\underline{b}) \cdot D\underline{f}(\underline{a}).$$

Proof. We know that

$$\underline{g}(\underline{b}+\underline{k}) - \underline{g}(\underline{k}) = D\underline{g}(\underline{b}) \cdot \underline{k} + \underline{E}_1(\underline{k})\|\underline{k}\|,$$

where $\underline{E}_1(k) \longrightarrow 0$ as $\underline{k} \longrightarrow \underline{0}$. Let us set $\underline{k} = \underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a})$ in this formula. Then

Thus the Jacobian matrix of $\underline{F}$ satisfies the matrix equation

$$D\underline{F}(\underline{a}) = D\underline{g}(\underline{b}) \cdot D\underline{f}(\underline{a}).$$

This is our generalized version of the chain rule.

There is, however, a problem here. We have just shown that if $\underline{f}$ and $\underline{g}$ are differentiable, then the partial derivatives of the composite function $\underline{F}$ exist. But we know that the mere <u>existence</u> of the partial derivatives of the function F_i is not enough to guarantee that F_i is differentiable. One needs to give a separate proof that if both $\underline{f}$ and $\underline{g}$ are differentiable, then so is the composite $\underline{F}(\underline{x}) = \underline{f}(\underline{g}(\underline{x}))$. (See Theorem 4 following.)

One can avoid giving a separate proof that $\underline{F}$ is differentiable by assuming a stronger hypothesis, namely that both $\underline{f}$ and $\underline{g}$ are <u>continuously</u> differentiable. In this case, the partials of $\underline{f}$ and $\underline{g}$ are continuous on their respective domains; then the formula

$$D_j F_i(\underline{x}) = \sum_{\ell=1}^{k} D_\ell g_i(\underline{f}(\underline{x})) \cdot D_j f_\ell(\underline{x}),$$

which we have proved, shows that $D_j F_i$ is also a continuous function of $\underline{x}$. Then by our basic theorem, F_i is differentiable for each i, so that $\underline{F}$ is differentiable, by definition.

We summarize these facts as follows:

<u>Theorem 3</u>. <u>Let</u> $\underline{f}$ <u>be defined on an open ball in</u> R^n <u>about</u> a, <u>taking values in</u> R^k; <u>let</u> $f(\underline{a}) = \underline{b}$. <u>Let</u> $\underline{g}$ <u>be defined in an open ball about</u> $\underline{b}$, <u>taking values in</u> R^p. <u>If</u> $\underline{f}$ <u>and</u> $\underline{g}$ <u>are continuously differentiable on their respective</u>

$$(**) \qquad \underline{g}(\underline{f}(\underline{a}+\underline{h})) - \underline{g}(\underline{f}(\underline{a})) = D\underline{g}(\underline{b}) \cdot (\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a}))$$

$$+ \underline{E}_1(\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a}))\|\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a})\| .$$

Now we know that

$$\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a}) = D\underline{f}(\underline{a}) \cdot \underline{h} + \underline{E}_2(\underline{h})\|\underline{h}\| ,$$

where $\underline{E}_2(\underline{h}) \longrightarrow \underline{0}$ as $\underline{h} \longrightarrow \underline{0}$. Plugging this into (**), we get the equation

$$\underline{g}(\underline{f}(\underline{a}+\underline{h}) - \underline{g}(\underline{f}(\underline{a})) = D\underline{g}(\underline{b}) \cdot D\underline{f}(\underline{a}) \cdot \underline{h} + D\underline{g}(\underline{b}) \cdot \underline{E}_2(\underline{h})\|\underline{h}\|$$

$$+ E_1(\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a}))\|\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a})\| .$$

Thus

$$\underline{F}(\underline{a}+\underline{h}) - \underline{F}(\underline{a}) = D\underline{g}(\underline{b}) \cdot D\underline{f}(\underline{a}) \cdot \underline{h} + \underline{E}_3(\underline{h})\|\underline{h}\| ,$$

where

$$\underline{E}_3(\underline{h}) = D\underline{g}(\underline{b}) \cdot \underline{E}_2(\underline{h}) + \underline{E}_1(\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a}))\frac{\|\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a})\|}{\|\underline{h}\|} .$$

We must show that $\underline{E}_3 \longrightarrow \underline{0}$ as $\underline{h} \longrightarrow \underline{0}$. The first term is easy, since $Dg(\underline{b})$ is constant and $\underline{E}_2(\underline{h}) \longrightarrow \underline{0}$ as $h \longrightarrow 0$. Furthermore, as $\underline{h} \longrightarrow \underline{0}$, the expression $\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a})$ approaches $\underline{0}$ (since $\underline{f}$ is continuous), so that

$\underline{E}_1(\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a})) \longrightarrow \underline{0}$. We need finally to show that the expression

$$\|\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a})\| / \|\underline{h}\|$$

is bounded as $\underline{h} \longrightarrow \underline{0}$. Then we will be finished. Now

$$\frac{\|\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a})\|}{\|\underline{h}\|} = \|D\underline{f}(\underline{a}) \cdot \frac{\underline{h}}{\|\underline{h}\|} + \underline{E}_2(\underline{h})\|$$

$$\leqslant \|D\underline{f}(\underline{a}) \cdot \underline{u}\| + \|\underline{E}_2(\underline{h})\|,$$

where $\underline{u}$ is a unit vector. Now $E_2(\underline{h}) \longrightarrow \underline{0}$ as $\underline{h} \longrightarrow \underline{0}$, and it is easy to see that $\|Df(\underline{a}) \cdot \underline{u}\| \leqslant nk \max|D_i f_j(\underline{a})|$. (Exercise!) Hence the expression $\|\underline{f}(\underline{a}+\underline{h}) - \underline{f}(\underline{a})\|/\|\underline{h}\|$ is bounded, and we are finished. □

Differentiating inverse functions.

Recall that if $f(x)$ is a differentiable real-valued function of a single real variable x, and if $f'(x) > 0$ for $a \leqslant x \leqslant b$, then f is strictly increasing, so it has an inverse g. Furthermore, g is differentiable and its derivative satisfies the formula

$$g'(f(x)) = \frac{1}{f'(x)} .$$

Part, but not all, of this theorem generalizes to vector functions. We shall show that *if* $\underline{f}$ has an inverse $\underline{g}$, and if $\underline{g}$ is differentiable, then there is a formula for $D\underline{g}$ analogous to this one. Specifically, we prove the following:

Theorem 5. *Let* S *be a subset of* R^n. *Suppose that* $\underline{f} : A \longrightarrow R^n$ *and that* $\underline{f}(\underline{a}) = \underline{b}$. *Suppose also that* $\underline{f}$ *has an inverse* $\underline{g}$.

If $\underline{f}$ *is differentiable at* $\underline{a}$, *and if* $\underline{g}$ *is differentiable at* $\underline{b}$, *then*

$$D\underline{g}(\underline{b}) = [D\underline{f}(\underline{a})]^{-1}.$$

Proof. Because $\underline{g}$ is inverse to $\underline{f}$, the equation $\underline{g}(\underline{f}(\underline{x})) = \underline{x}$ holds for all $\underline{x}$ in S. Now both $\underline{f}$ and $\underline{g}$ are differentiable and so is the composite function $\underline{g}(\underline{f}(\underline{x}))$. Thus we can use the chain rule to compute

$$D\underline{g}(\underline{b}) \cdot D\underline{f}(\underline{a}) = D(\text{identity}) = I_n.$$

Since the matrices involved are n by n, this equation implies that

$$Dg(\underline{b}) = [Df(\underline{a})]^{-1}. \qquad \square$$

Remark 1. This theorem shows that in order for the differentiable function $\underline{f}$ to have a differentiable inverse, it is *necessary* that the Jacobian matrix $D\underline{f}(\underline{a})$ have rank n. Roughly speaking, this condition is also *sufficient* for $\underline{f}$ to have an inverse.

More precisely, one has the following result, which is the famous "Inverse Function Theorem" of Analysis:

Suppose $\underline{f}$ is defined and continuously differentiable in an open ball of R^n about $\underline{a}$, taking values in R^n. If $D\underline{f}(\underline{a})$ has rank n, then there is some (probably smaller) open ball B about $\underline{a}$, such that $\underline{f}$ carries B in a 1-1 fashion onto an open set C in R^n. Furthermore, the inverse function $\underline{g} : C \longrightarrow B$ is continuously differentiable, and $D\underline{g}(\underline{f}(\underline{x})) = [D\underline{f}(\underline{x})]^{-1}$.

Remark 2. For a function of a single variable, $y = f(x)$, the rule for the derivative of the inverse function $x = g(y)$ is often written in the form

$$\frac{dx}{dy} = \frac{1}{dy/dx} .$$

This formula is easy to remember; the Leibnitz notation for derivatives "does the work for you". It is tempting to think that a similar result should hold for a function of several variables. It does not.

For example, suppose

$$\underline{f}(x,y) = (u,v)$$

is a differentiable transformation from the $x - y$ plane to the $u - v$ plane. And suppose it has an inverse; given by

$$(x,y) = \underline{g}(u,v).$$

Our theorem tells us that if $\underline{f}(\underline{a}) = \underline{b}$, then

$$D\underline{g}(\underline{b}) = [D\underline{f}(\underline{a})]^{-1}.$$

If we write out these matrices in Leibnitz notation, we obtain the equation

$$\begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix} = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix}^{-1}.$$

Now the formula for the inverse of a matrix gives (in the case of a 2 by 2 matrix) the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Applying this formula, we obtain the equation

$$\begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix} = \frac{1}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}\begin{bmatrix} \partial v/\partial y & -\partial u/\partial y \\ -\partial v/\partial x & \partial u/\partial x \end{bmatrix}.$$

This means, for example, that

$$\frac{\partial x}{\partial v} = \frac{1}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}\left(- \frac{\partial u}{\partial y}\right).$$

Thus the simplistic idea that $\partial x/\partial v$ "should be" the reciprocal of $\partial v/\partial x$ is very far from the truth. The Leibnitz notation simply doesn't "do the work for you" in dimensions greater than 1. Matrix notation does.

Implicit differentiation.

Suppose $\underline{F}$ is a function from R^{n+k} to R^n; let us write it in the form

$$\underline{F}(\underline{x},\underline{y}) = \underline{F}(x_1,\ldots,x_n,y_1,\ldots,y_k).$$

Let $\underline{c}$ be a point of R^n, and consider the equation

$$\underline{F}(\underline{x},\underline{y}) = \underline{c}.$$

This equation represents a system of n equations in $n + k$ unknowns. In general, we would expect to be able to solve this system for $\underline{n}$ of the unknowns in terms of the others. For instance, in the present case we would expect to be able to solve this system for $\underline{x}$ in terms of $\underline{y}$. We would also expect the resulting function $\underline{x} = \underline{g}(\underline{y})$ to be differentiable.

Assuming this expectation to be correct, one can then calculate the derivative of the resulting function $\underline{g}$ by using the chain rule. One understands best how this is done by working through a number of examples. Apostol works several in sections 9.6 and 9.7. At this point, you should read 9.6 and Examples 1,2,3, and 6 of 9.7.

A natural question to ask now is the following: to what extent our assumptions are correct, that the given equation determines $\underline{x}$ as a function of $\underline{y}$. We discuss that question now.

First let us consider the problem discussed on p. 294 of the text. It involves an equation of the form

$$F(x,y,z) = 0 ,$$

where F is continuously differentiable. Assuming that one can in theory solve this equation for z as a function of x and y, say $z = f(x,y)$, Apostol derives equations for the partials of this unknown function:

$$\frac{\partial f}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} .$$

Here the functions on the right side of these equations are evaluated at the point $(x,y,f(x,y))$.

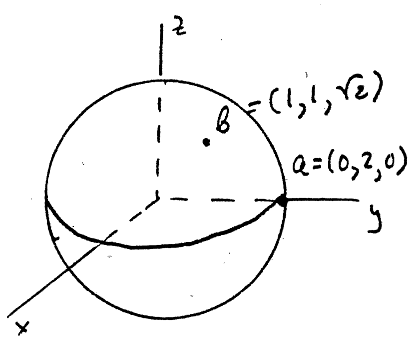
Note that it was <u>necessary</u> to assume that $\partial F/\partial z \neq 0$, in order to carry out these calculations. It is a remarkable fact that the condition $\partial F/\partial z \neq 0$ is also <u>sufficient</u> to justify the assumptions we made in carrying them out. This is a consequence of a famous theorem of Analysis called the Implicit Function Theorem. One consequence of this theorem is the following: If one has a point (x_0,y_0,z_0) that satisfies the equation $F(x,y,z) = 0$, and if $\partial F/\partial z \neq 0$ at this point, then there exists a unique differentiable function $f(x,y)$, defined in an open set B about (x_0,y_0), such that $f(x_0,y_0) = z_0$ and such that

$$F(x,y,f(x,y)) = 0$$

for all (x,y) in B. Of course, once one knows that f exists and is differentiable, one can find its partials by implicit differentiation, as explained in the text.

Example 1. Let $F(x,y,z) = x^2 + y^2 + z^2 + 1$. The equation $F(x,y,z) = 0$ cannot be solved for z in terms of x and y; for in fact there is no point that satisfies the equation.

Example 2. Let $F(x,y,z) = x^2 + y^2 + z^2 - 4$. The equation $F(x,y,z) = 0$ is satisfied by the point $a = (0,2,0)$. But $\partial F/\partial z = 0$ at the point a, so the implicit function theorem does not apply. This fact is hardly surprising, since it is clear from the picture that z is not determined as a function of (x,y) in an open set about the point $(x_0,y_0) = (0,2)$.



However, the point $b = (1,1,\sqrt{2})$ satisfies the equation also, and $\partial F/\partial z \neq 0$ at this point. The implicit function theorem implies that there is a function $f(x,y)$ defined in a neighborhood of $(x_0,y_0) = (1,1)$ such that $f(1,1) = \sqrt{2}$ and f satisfies the equation $F(x,y,z) = 0$ identically.

Note that f is not uniquely determined unless we specify its value at (x_0,y_0). There are two functions f defined in a neighborhood of $(1,1)$ that satisfy the equation $f(x,y,z) = 0$, namely,

$$z = [4 - x^2 - y^2]^{\frac{1}{2}} \quad \text{and} \quad z = -[4 - x^2 - y^2]^{\frac{1}{2}} .$$

However, only one of them satisfies the condition $f(1,1) = \sqrt{2}$.

Note that at the point $a = (0,2,0)$ we do have the condition $\partial F/\partial y \neq 0$. Then the implicit function theorem implies that y is determined as a function of (x,z) near this point. The picture makes this fact clear.

Now let us consider the more general situation discussed on p. 296 of the text. We have two equations

$$(*)\qquad \begin{aligned} F(x,y,z,w) &= 0 \\ G(x,y,z,w) &= 0 \end{aligned}$$

where F and G are continuously differentiable. (We have inserted an extra variable to make things more interesting.) Assuming there are functions $x = X(z,w)$ and $y = Y(z,w)$ that satisfy these equations for all points in an open set in the (z,w) plane, we have the identities

$$F(X,Y,z,w) = 0 \quad \text{and} \quad G(X,Y,z,w) = 0 ,$$

whence (differentiating with respect to z),

$$\frac{\partial F}{\partial x}\frac{\partial X}{\partial z} + \frac{\partial F}{\partial y}\frac{\partial Y}{\partial z} + \frac{\partial F}{\partial z} = 0 ,$$

$$\frac{\partial G}{\partial x}\frac{\partial X}{\partial z} + \frac{\partial G}{\partial y}\frac{\partial Y}{\partial z} + \frac{\partial G}{\partial z} = 0 .$$

These are linear equations for $\partial X/\partial z$ and $\partial Y/\partial z$; we can solve them if the coefficient matrix

$$\begin{bmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{bmatrix} = \frac{\partial F,G}{\partial x,y}$$

is non-singular. One can use Cramer's rule, as in the text, or one can write the solution in the form

$$\begin{bmatrix} \partial X/\partial z \\ \partial Y/\partial z \end{bmatrix} = -\left(\frac{\partial F,G}{\partial x,y}\right)^{-1} \begin{bmatrix} \partial F/\partial z \\ \partial G/\partial z \end{bmatrix} .$$

The functions on the right side of this equation are evaluated at the point $(X(z,w),Y(z,w),z,w)$, so that both sides of the equation are functions of z and w alone.

You can check that one obtains an equation for the other partials of X and Y if one replaces z by w throughout.

All this is discussed in the text. But now let us note that in order to carry out these calculations, it was <u>necessary</u> to assume that the matrix $\partial F,G/\partial x,y$ was non-singular. Again, it is a remarkable fact that this condition is also <u>sufficient</u> to justify the assumptions we have made. Specifically, the Implicit Function Theorem tells us that if (x_0,y_0,z_0,w_0) is a point satisfying the equations (*), and if the matrix $\partial F,G/\partial x,y$ is non-singular at this point, then there do exist unique differentiable functions $X(z,w)$ and $Y(z,w)$ defined in an open set about (z_0,w_0), such that

$$X(z_0,w_0) = x_0 \quad \text{and} \quad Y(z_0,w_0) = y_0 ,$$

and such that F and G vanish identically when X and Y are substituted for x and y. Thus under this assumption all our calculations are justified.

<u>Example</u> 3. Consider the equations

$$F(x,y,z,w) = 3x^2z + 6wy^2 - 2z + 1 = 0 ,$$

$$G(x,y,z,w) = xz - 4y/z - 3w - 7 = 0 .$$

The points $(x_0,y_0,z_0,w_0) = (1,2,-1,0)$ and $(x_1,y_1,z_1,w_1) = (1,\frac{1}{2},2,-2)$ satisfy these equations, as you can check. We calculate

$$\partial F,G/\partial x,y = \begin{bmatrix} 6xz & 12wy \\ z & -4/z \end{bmatrix} .$$

At the point (x_0,y_0,z_0,w_0), this matrix equals $\begin{bmatrix} -6 & 0 \\ -1 & 4 \end{bmatrix}$, which is non-singular. Therefore, there exist unique functions $x = X(z,w)$ and $y = Y(z,w)$, defined in a neighborhood of $(z_0,w_0) = (-1,0)$ that satisfy these equations identically, such that $X(-1,0) = 1$ and $Y(-1,0) = 2$.

Since we know the values of X and Y at the point $(-1,0)$, we can find the values of their partial derivatives at this point also. Indeed,

$$\begin{bmatrix} \partial X/\partial z \\ \partial Y/\partial z \end{bmatrix} = - \begin{bmatrix} 6Xz & 12wY \\ z & -4/z \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3X^2 - 2 \\ X + 4Y/z^2 \end{bmatrix}$$

$$= - \begin{bmatrix} 6 & 0 \\ -1 & 4 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 9 \end{bmatrix} = - \frac{1}{24} \begin{bmatrix} 4 \\ 55 \end{bmatrix} .$$

On the other hand, at the point $(x_1,y_1,z_1,w_1) = (1,\frac{1}{2},2,-2)$ the matrix $\partial F,G/\partial x,y$ equals

$$\begin{bmatrix} 12 & -12 \\ 2 & -2 \end{bmatrix},$$

which is singular. Therefore we do not expect to be able to solve for x and y in terms of z and w near this point. However, at this point, we have

$$\partial F,G/\partial x,w = \begin{bmatrix} 6xz & 6y^2 \\ z & -3 \end{bmatrix} = \begin{bmatrix} 12 & 3/2 \\ 2 & -3 \end{bmatrix} .$$

Therefore, the implicit function theorem implies that we <u>can</u> solve for x and w in terms of y and z near this point.

<u>Exercises</u>

1. Given the continuously differentiable scalar field $f(x,y)$, let $\phi(t) = f(t^2,t^3 + 1)$. Find $\phi'(1)$, given that $\vec{\nabla} f(1,2) = 5\vec{i} - \vec{j}$.

2. Find the point on the surface $z = xy$ nearest the point $(2,2,0)$.

3. A rectangular box, open at the top, is to hold 256 cubic inches. Find the dimensions that minimize surface area.

4. Find parametric equations for the tangent line to the curve of intersection of the surfaces

$$x^2 + y^2 + 2z^2 = 13,$$

$$z = x^2 - xy^3 \;,$$

at the point $(2,1,2)$.

5. Let f be a scalar function of 3 variables. Define

$$F(t) = f(3t^2, 2t+1, 3-t^3).$$

Express $F'(1)$ in terms of the first order partials of f at the point $(3,3,2)$.

Express $F''(1)$ in terms of the first and second order partials of f at the point $(3,3,2)$.

6. Let $\underline{f} : R^2 \longrightarrow R^2$ and let $\underline{g} : R^2 \longrightarrow R^3$. Suppose that

$\underline{f}(0,0) = (1,2)$ $\quad$ $\underline{f}(1,2) = (0,0)$.

$\underline{g}(0,0) = (1,3,2)$ $\quad$ $\underline{g}(1,2) = (-1,0,1)$.

Suppose that

$$D\underline{f}(0,0) = \begin{bmatrix} -1 & 2 \\ 6 & 3 \end{bmatrix} \qquad D\underline{f}(1,2) = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$D\underline{g}(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad D\underline{g}(1,2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}.$$

a) If $\underline{h}(\underline{x}) = \underline{g}(\underline{f}(\underline{x}))$, find $D\underline{h}(0,0)$.

b) If $\underline{f}$ has an inverse $\underline{k} : R^2 \longrightarrow R^2$, find $D\underline{k}(0,0)$.

7. Consider the functions of Example 3. Find the partials $\partial X/\partial w$ and $\partial Y/\partial w$ at the point $(z_0, w_0) = (-1, 0)$.

8. For the functions F and G of Example 3, compute $\partial(F,G)/\partial(x,y)$ at the point $(1,\frac{1}{2},2,-2,)$. Given the equations $F = 0$, $G = 0$, for which pairs of variables is it possible to solve in terms of the other two near this point?

The second-derivative test for extrema of a function of two variables.

Theorem. *Suppose that* $f(x_1,x_2)$ *has continuous second-order partial derivatives in a ball* B *about* a. *Suppose that* D_1f *and* D_2f *vanish at* a. *Let*

$$A = D_{1,1}f(\underline{a}), \quad B = D_{1,2}f(\underline{a}), \quad C = D_{2,2}f(\underline{a}).$$

(a) *If* $B^2 - AC > 0$, *then* f *has a saddle point at* a.

(b) *If* $B^2 - AC < 0$ *and* $A > 0$, *then* f *has a relative minimum at* a.

(c) *If* $B^2 - AC < 0$ *and* $A < 0$, *then* f *has a relative maximum at* a.

(d) *If* $B^2 - AC = 0$, *the test is inconclusive*.

Proof. *Step 1.* We first prove a version of Taylor's theorem with remainder for functions of two variables:

Suppose $f(x_1,x_2)$ has continuous second-order partials in a ball B centered at $\underline{a}$. Let $\underline{v}$ be a fixed vector; say $\underline{v} = (h,k)$. Then

$$(*)\quad \begin{aligned} f(\underline{a}+t\underline{v}) &= f(\underline{a}) + [D_1f(\underline{a})\cdot h + D_2f(\underline{a})\cdot k]t \\ &+ \{D_{1,1}f(\underline{a}^*)h^2 + 2D_{1,2}f(\underline{a}^*)hk + D_{2,2}f(\underline{a}^*)k^2\}\frac{t^2}{2}, \end{aligned}$$

where $\underline{a}^*$ is some point on the line segment from $\underline{a}$ to $\underline{a} + t\underline{v}$.

We derive this formula from the single-variable form of Taylor's theorem. Let $g(t) = f(\underline{a}+t\underline{v})$, i.e., Let $F(t\underline{v})$ denote the left side of this equation. We will be concerned about the sign of $F(t\underline{v})$ when t is small, because that sign will depend on whether f has a local maximum or a local minimum at $\underline{a}$, or neither.

Step 3. From now on, let $\underline{v} = (h,k)$ be a unit vector. Consider the quadratic function

$$Q(\underline{v}) = Q(h,k) = Ah^2 + 2Bhk + Ck^2.$$

We shall determine what values Q takes as $\underline{v}$ varies over the unit circle.

Case 1. If $B^2 - AC < 0$, then we show that $Q(\underline{v})$ has the same sign as A, for all unit vectors $\underline{v}$.

Proof. When $\underline{v} = (1,0)$, then $Q(\underline{v}) = A$; thus $Q(\underline{v})$ has the same sign as A in this case. Consider the continuous function $Q(\cos t, \sin t)$. As t ranges over the interval $[0,2\pi]$, the vector $(\cos t, \sin t)$ ranges over all unit vectors in V_2. If this function takes on a value whose sign is different from that of A, then by the intermediate-value theorem, there must be a t_o such that $Q(\cos t_o, \sin t_o) = 0$. That is,

$$Q(h_o,k_o) = 0$$

for some unit vector (h_o,k_o). Now if $h_o \neq 0$, this means that the number k_o/h_o is a real root of the equation

$$A + 2Bx + Cx^2 = 0.$$

$$g(t) = f(a_1 + th,\ a_2 + tk).$$

We know that $g(t) = g(0) + g'(0)\cdot t + g''(c)\cdot t^2/2!$ where c is between 0 and t. Calculating the derivatives of g gives

$$g'(t) = D_1 f(a_1+th, a_2+tk)\cdot h + D_2 f(a_1+th, a_2+tk)\cdot k,$$

$$g''(t) = (D_{1,1}f)h^2 + (D_{1,2}f)hk + (D_{2,1}f)kh + (D_{2,2}f)k^2,$$

from which formula (*) follows. Here $\underline{a}^* = \underline{a} + c\underline{v}$, where c is between 0 and t.

<u>Step 2</u>. In the present case, the first partials of f vanish at $\underline{a}$, so that

$$f(\underline{a}+t\underline{v}) - f(\underline{a}) \sim \{Ah^2 + 2Bhk + Ck^2\}t^2/2.$$

The only reason this is an approximation rather than an equality is that the second partials are evaluated at the unknown point $\underline{a}^*$ instead of at $\underline{a}$. This matter will be disposed of by using elementary epsilonics. Formally, we have the equation

$$(**)\quad \begin{aligned} \frac{2}{t^2}[f(\underline{a}+t\underline{v}) - f(\underline{a})] &= \{Ah^2 + 2Bhk + Ck^2\} \\ &+ [D_{1,1}f(\underline{a}^*)-A]h^2 + 2[D_{1,2}f(\underline{a}^*)-B]hk + [D_{2,2}f(\underline{a}^*)-C]k^2. \end{aligned}$$

Note that the last three terms are small if $\underline{a}^*$ is close to $\underline{a}$, because the second partials are continuous.

But this equation has a real root only if $B^2 - AC > 0$. Similarly, if $k_o \neq 0$, the number h_o/k_o is a real root of the equation

$$Ax^2 + 2Bx + C = 0;$$

again we conclude that $B^2 - AC > 0$. Thus in either case we are led to a contradiction.

<u>Case 2</u>. If $B^2 - AC > 0$, then we show that $Q(\underline{v})$ takes on both positive and negative values.

<u>Proof</u>. If $A \neq 0$, the equation $Ax^2 + Bx + C = 0$ has two distinct real roots. Thus the equation $y = Ax^2 + 2Bx + C$ represents a parabola that crosses the x-axis at two distinct points. On the other hand, if $A = 0$, then $B \neq 0$ (since $B^2 - AC > 0$); in this case the equation $y = Ax^2 + 2Bx + C$ represents a line with non-zero slope. It follows that in either case, there is a number x_o for which

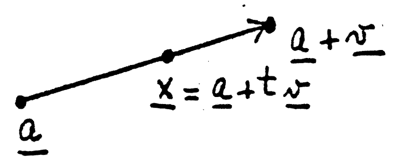
$$Ax_o^2 + 2Bx_o + C < 0,$$

and a number x_1 for which

$$Ax_1^2 + 2Bx_1 + C > 0.$$

Let (h_o,k_o) be a unit vector with $h_o/k_o = x_o$ and let (h_1,k_1) be a unit vector with $h_1/k_1 = x_1$. Then $Q(h_o,k_o) < 0$ and $Q(h_1,k_1) > 0$.

<u>Step 4</u>. We prove part (a) of the theorem. Assume $B^2 - AC > 0$. Let $\mathbf{v}_o$ be a unit vector for which $Q(\mathbf{v}_o) > 0$. Examining formula (**), we see that the expression



$2[f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a})]/t^2$ approaches $Q(v_o)$ as $t \longrightarrow 0$. Let $\mathbf{x} = \mathbf{a} + t\mathbf{v}$ and let $t \longrightarrow 0$. Then $\mathbf{x}$ approaches $\mathbf{a}$ along the straight line from $\mathbf{a}$ to $\mathbf{a} + \mathbf{v}_o$, and the expression $f(\mathbf{x}) - f(\mathbf{a})$ approaches zero through positive values. On the other hand, if $\mathbf{v}_1$ is a point at which $Q(\mathbf{v}_1) < 0$, then the same argument shows that as $\mathbf{x}$ approaches $\mathbf{a}$ along the straight line from $\mathbf{a}$ to $\mathbf{a} + \mathbf{v}_1$, the expression $f(\mathbf{x}) - f(\mathbf{a})$ approaches $\mathbf{0}$ through negative values.

We conclude that f has a saddle point at $\mathbf{a}$.

<u>Step 5</u>. We prove parts (b) and (c) of the theorem. Examining equation (**) once again. We know that $|Q(\mathbf{v})| > 0$ for all unit vectors $\mathbf{v}$. Then $|Q(\mathbf{v})|$ has a positive minimum m, as $\mathbf{v}$ ranges over all unit vectors. (Apply the extreme-value theorem to the continuous function $|Q(\cos t, \sin t)|$, for $0 \leq t \leq 2\pi$.) Now choose δ small enough that each of the three square-bracketed expressions on the right side of (**) is less than $m/3$ whenever $\mathbf{a}^*$ is within δ of $\mathbf{a}$. Here we use continuity of the second-order partials. If $0 < t < \delta$, then $\mathbf{a}^*$ is on the line from $\mathbf{a}$ to $\mathbf{a} + \delta\mathbf{v}$; since $\mathbf{v}$ is a unit vector, then the right side of (*) has the same sign as A whenever $0 < t < \delta$. If $A > 0$, this means that $f(\mathbf{x}) - f(\mathbf{a}) > 0$ whenever

$0 < |\underline{x}-\underline{a}| < \delta$, so f has a relative minimum at $\underline{a}$. If $A < 0$, then $f(\underline{x}) - f(\underline{a}) < 0$ whenever $0 < |\underline{x}-\underline{a}| < \delta$, so f has a relative maximum at $\underline{a}$.

For examples illustrating (d), see the exercises. □

Exercises

1. Show that the function $x^4 + y^4$ has a relative minimum at the origin, while the function $x^4 - y^4$ has a saddle point there. Conclude that the second-derivative test is inconclusive if $B^2 - AC = 0$.

2. Use Taylor's theorem to derive the second derivative test for maxima and minima of a function $f(x)$ of a single variable. If $f'(a) = f''(a) = 0$ and $f'''(a) \neq 0$, what can you say about the existence of a relative maximum or minimum at f at a?

3. Suppose $f(x)$ has continuous derivatives of orders $1,\ldots,n+1$ near $x = a$. Suppose

$$f'(a) = f''(a) = \cdots = f^{(n)}(a) = 0$$

and $f^{(n+1)}(a) \neq 0$. What can you say about the existence of a relative maximum or minimum of f at a? Prove your answer correct.

4. (a) Suppose $f(x_1,x_2)$ has continuous third-order partials near $\underline{a}$. Derive a third-order version of formula (*) of the preceding theorem.

(b) Derive the general version of Taylor's theorem for functions of two variables.

[The following "operator notation" is convenient

$$(hD_1+kD_2)f\big|_{\underline{x}=\underline{a}} = hD_1f(\underline{a}) + kD_2f(\underline{a}),$$

$$(hD_1+kD_2)^2f\big|_{\underline{x}=\underline{a}} = h^2D_1D_1f(\underline{a}) + 2hkD_1D_2f(\underline{a}) + h^2D_2D_2f(\underline{a}),$$

and similarly for $(hD_1+kD_2)^n$.]

The extreme-value theorem and the small-span theorem.

The proofs of the extreme-value theorem and small-span theorem for rectangles given in Apostol are sufficiently condensed to cause some students difficulty. Here are the details. We shall prove the theorems only for R^2, but the proofs go through without difficulty in R^n.

A rectangle Q in R^2 is the Cartesian product of two closed intervals $[a,b]$ and $[c,d]$;

$$Q = [a,b] \times [c,d] = \{(x,y) \mid a \leq x \leq b \text{ and } c \leq y \leq d\}.$$

The intervals $[a,b]$ and $[c,d]$ are called the component intervals of Q. If

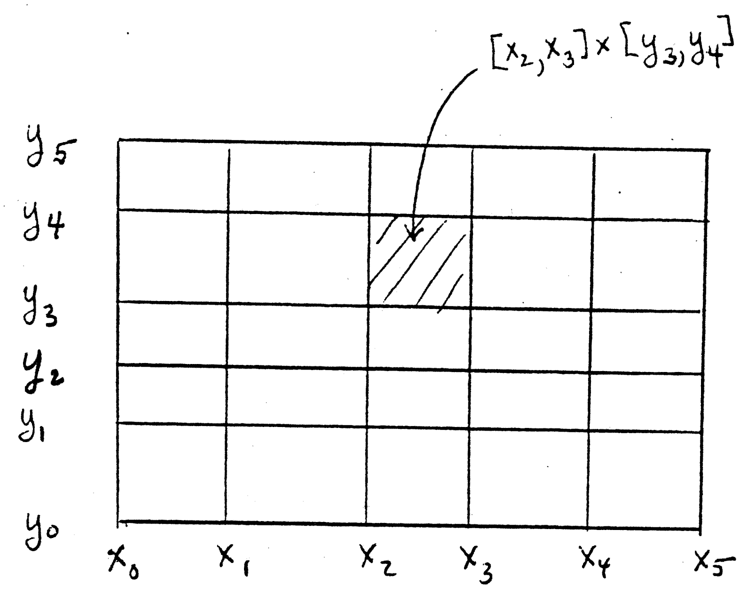
$$P_1 = \{x_0, x_1, \ldots, x_n\}$$

is a partition of $[a,b]$, and if

$$P_2 = \{y_0, y_1, \ldots, y_m\}$$

is a partition of $[c,d]$, then the cartesian product $P_1 \times P_2$ is said to be a partition of Q. Since P_1 partitions $[a,b]$ into n subintervals and P_2 partitions $[c,d]$ into m subintervals, the partition $P = P_1 \times P_2$ partitions Q into mn subrectangles, namely the rectangles

$$[x_{i-1}, x_i] \times [y_{j-1}, y_j].$$



Theorem (small-span theorem). Let f be a scalar function that is continuous on the rectangle

$$Q = [a,b] \times [c,d]$$

in R^2. Then, given $\epsilon > 0$, there is a partition of Q such that f is bounded on every subrectangle of the partition and such that the span of f in every subrectangle of the partition is less than ϵ.

Proof. For purposes of this proof, let us use the following terminology: If Q_0 is any rectangle contained in Q, let us say that a partition of Q_0 is "ϵ-nice" if f is bounded on every subrectangle R of the partition and if the span of f in every subrectangle of the partition is less than ϵ. We recall that the span of f in the set S is defined by the equation

$$\text{span}_S f = \sup\{f(x) \mid x \in S\} - \inf\{f(x) \mid x \in S\}.$$

Recall also that if S_1 is a subset of S, then

$$\operatorname{span}_{S_1} f \leq \operatorname{span}_S f.$$

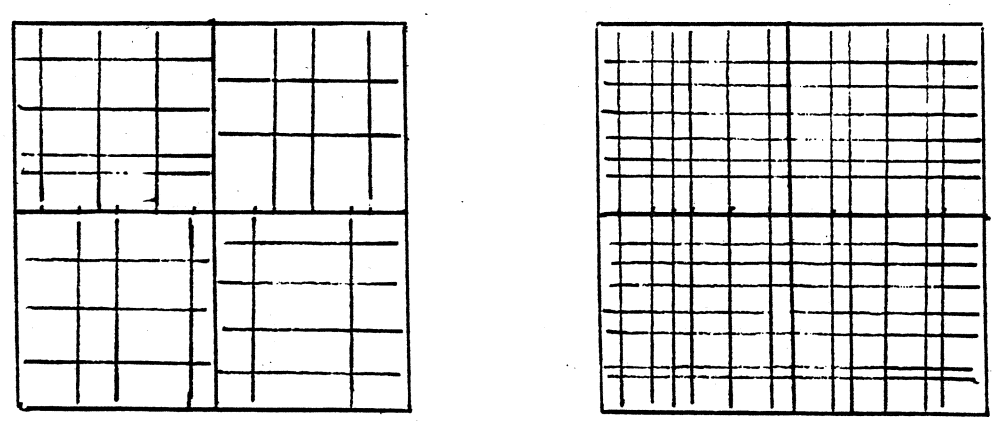
To begin, we note the following elementary fact: Suppose

$$Q_0 = [a_0,b_0] \times [c_0,d_0]$$

is any rectangle contained in Q. Let us bisect the first component interval $[a_0,b_0]$ of Q_0 into two subintervals $I_1 = [a_0,p]$ and $I_2 = [p,b_0]$, where p is the midpoint of $[a_0,b_0]$. Similarly, let us bisect $[c_0,d_0]$ into two subintervals J_1 and J_2. Then Q_0 is written as the union of the four rectangles

$$I_1 \times J_1 \quad \text{and} \quad I_2 \times J_1 \quad \text{and} \quad I_1 \times J_2 \quad \text{and} \quad I_2 \times J_2.$$

Now if each of these rectangles has a partition that is ϵ_0-nice, then we can put these partitions together to get a partition of Q_0 that is ϵ_0-nice. The figure indicates the proof; each of the subrectangles of the new partition is contained in a subrectangle of one of the old partitions.



Now we prove the theorem. We suppose the theorem is false and derive a contradiction. That is, we assume that for some $\epsilon_0 > 0$, the rectangle Q has no partition that is ϵ_0-nice.

Let us bisect each of the component intervals of Q, writing Q as the union of four rectangles. Not all of these smaller rectangles have partitions that are ϵ_0-nice, for if they did, then Q would have such a partition. Let Q_1 be one of these smaller rectangles, chosen so that Q_1 does not have a partition that is ϵ_0-nice.

Now we repeat the process. Bisect each component interval of Q_1 into four smaller rectangles. At least one of these smaller rectangles has no partition that is ϵ_0-nice; let Q_2 denote one such.

Continuing similarly, we obtain a sequence of rectangles

$$Q, Q_1, Q_2, \ldots$$

none of which have partitions that are ϵ_0-nice. Consider the left-hand end points of the first component interval of each of these rectangles. Let s be their least upper bound. Similarly, consider the left-hand end points of the second component interval of each of these rectangles, and let t be their least upper bound. Then the point (s,t) belongs to all of the rectangles $Q, Q_1, Q_2, \ldots$.

Now we use the fact that f is continuous at the point (s,t). We choose a ball of radius r centered at (s,t) such that the span of f in this ball is less than ϵ_0. Because the rectangles Q_m become arbitrarily small as m increases, and because they all contain the point (s,t), we can choose m large enough that Q_m lies within this ball.

Now we have a contradiction. Since Q_m is contained in the ball of radius r centered at (s,t), the span of f in Q_m is less than ϵ_0. But this implies that there _is_ a partition of Q_m that is ϵ_0-nice, namely the trivial partition! Thus we have reached a contradiction. □

Corollary. _Let_ f _be a scalar function that is continuous on the rectangle_ Q. _Then_ f _is bounded on_ Q.

Proof. Set $\epsilon = 1$, and choose a partition of Q that is ϵ-nice. This partition divides Q into a certain number of subrectangles, say $R_1,\ldots,R_{mn}$. Now f is bounded on each of these subrectangles, by hypothesis; say

$$|f(\underline{x})| \leq M_i \quad \text{for } \underline{x} \in R_i.$$

Then if $M = \max\{M_1,\ldots,M_{mn}\}$, we have

$$|f(\underline{x})| \leq M$$

for all $\underline{x} \in Q$. □

Theorem (extreme-value theorem). _Let_ f _be a scalar function that is continuous on the rectangle_ Q. _Then there are points_ x_0 _and_ x_1 _of_ Q _such that_

$$f(\underline{x}_0) \leq f(\underline{x}) \leq f(\underline{x}_1)$$

for all $x \in Q$.

Proof. We know f is bounded on Q; let

$$M = \sup\{f(\underline{x}) \mid \underline{x} \in Q\}.$$

We wish to show there is a point $\underline{x}_1$ of Q such that $f(\underline{x}_1) = M$.

Suppose there is no such a point. Then the function $M - f(\underline{x})$ is continuous and positive on Q, so that the function

$$g(\underline{x}) = \frac{1}{M - f(\underline{x})}$$

is also continuous and positive on Q. By the preceding corollary g is bounded on Q; let C be a positive constant such that $g(\underline{x}) \leq C$ for $\underline{x} \in Q$. Then

$$\frac{1}{M - f(\underline{x})} \leq C, \text{ or}$$

$$f(\underline{x}) \leq M - (1/C)$$

for all $\underline{x}$ in Q. Then $M - (1/C)$ is an upper bound for the set of values of $f(\underline{x})$ for $\underline{x}$ in Q, contradicting the fact that M is the least upper bound for this set.

A similar argument proves the existence of a point $\underline{x}_0$ of Q such that

$$f(\underline{x}_0) = \inf\{f(\underline{x}) \mid \underline{x} \in Q\}. \qquad \square$$

Exercises on line integrals

1. Find the centroid of a homogeneous wire in shape of the parabolic arc

$$y = x^2 \quad \text{for} \quad -1 \le x \le 1.$$

[Use a table of integrals if you wish.]

2. Let

$$\underline{f}(x,y) = \frac{-y\underline{i}+x\underline{j}}{x^2+y^2}.$$

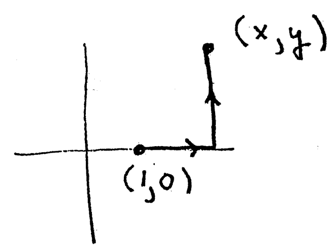
on the set S consisting of all $(x,y) \neq 0$.

(a) Show that $D_2 f_1 = D_1 f_2$ on S.

(b) Compute the line integral $\int_C \underline{f} \cdot d\underline{\alpha}$ from $(a,0)$ to $(-a,0)$ when C is the upper half of the circle $x^2 + y^2 = a^2$. Compute it when C is the lower half of the same circle.

3. Let $\underline{f}$ be as in problem 2. Let U be the set of all (x,y) with $x > 0$. Find a potential function for $\underline{f}$ that is defined in U. [Hint: Set

$$\phi(x,y) = \int_C \underline{f} \cdot d\underline{\alpha} \quad \text{where } C \text{ is the curve}$$



4. Let $\underline{f}$ be a continuous vector field defined in the open, connected subset S of R^n. Suppose that $\underline{f} = \vec{\nabla}\phi_1$ and $\underline{f} = \vec{\nabla}\phi_2$ in S. Show that $\phi_1 - \phi_2$ is a constant function. [Hint: Apply Thoerem 10.3.]