

**MODEL ANSWERS TO HWK #4  
(18.022 FALL 2010)**

- (1) (i)  $f$  is nowhere continuous. Let  $\epsilon = \frac{1}{2}$ . If  $x$  is rational, since irrational numbers are dense in reals, for any  $\delta > 0$  there is irrational  $y$  such that  $|y - x| < \delta$ , and  $|f(y) - f(x)| = 1 > \epsilon$ . Therefore  $f$  is not continuous at  $x$ . If  $x$  is irrational, similarly from that rational numbers are dense in reals, for any  $\delta > 0$  there is rational  $y$  such that  $|y - x| < \delta$ , and  $|f(y) - f(x)| = 1 > \epsilon$ . Therefore  $f$  is not continuous at  $x$ .
- (ii)  $f$  is continuous only at  $x = 0$ . If  $x \neq 0$ , then let  $\epsilon = \frac{|x|}{2}$ . Now by the same argument as in (i), for any  $\delta > 0$  there is  $y$  such that  $|f(y) - f(x)| = |x| > \epsilon$ . Therefore  $f$  is not continuous at  $x \neq 0$ . If  $x = 0$ , then for any  $\epsilon > 0$ ,  $|f(y) - f(x)| = |f(y)| < \epsilon$  for  $y$  such that  $|y - x| = |y| < \frac{\epsilon}{2}$ . Therefore  $f$  is continuous at  $x = 0$ .
- (2) (i) If  $f$  is continuous at  $x$  then  $\forall \epsilon > 0, \exists \delta > 0$  such that  $\forall \|y - x\| < \delta, \|f(y) - f(x)\| < \epsilon$ . Since  $\|f(y) - f(x)\| = \sqrt{(f_1(y) - f_1(x))^2 + \cdots + (f_m(y) - f_m(x))^2} > |f_i(y) - f_i(x)|$  for all  $i = 1, \dots, m$ ,  $f_i$  is also continuous at  $x$  for all  $i$ .
- (ii) If  $f_i$ 's are all continuous at  $x$ , then  $\forall \epsilon > 0$  and  $\forall i = 1, \dots, m, \exists \delta_i > 0$  such that  $\forall \|y - x\| < \delta_i, |f_i(y) - f_i(x)| < \frac{\epsilon}{\sqrt{m}}$ . Let  $\delta = \min(\delta_1, \dots, \delta_m)$ . We get  $\forall \|y - x\| < \delta, \|f(y) - f(x)\| = \sqrt{(f_1(y) - f_1(x))^2 + \cdots + (f_m(y) - f_m(x))^2} < \epsilon$ . Therefore  $f$  is continuous at  $x$ .
- (3) (a) By definition,

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = 0,$$

$$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = 0$$

Therefore both partial derivatives exist at  $(0, 0)$ . Now

$$h(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 0$$

Hence,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x, y) - h(x, y)}{\|(x, y) - (0, 0)\|} = \lim_{(x,y) \rightarrow (0,0)} \frac{|xy|}{\sqrt{x^2 + y^2}} = \lim_{r \rightarrow 0} \frac{r^2 |\sin \theta \cos \theta|}{r} = 0,$$

and  $f(x, y)$  is differentiable at  $(0, 0)$ .

- (b) By the symmetry between  $x$  and  $y$ , it's enough to prove the claim for  $f_x(x, y)$ . We have,

$$f_x(a, b) = |b| \lim_{h \rightarrow 0} \frac{|a + h| - |a|}{h} = \begin{cases} |b|, & a > 0 \\ -|b|, & a < 0 \end{cases}$$

So  $f_x$  is not continuous at  $(0, b)$  if  $|b| > 0$ . For any neighborhood of the origin, we can choose such point  $(0, b)$ , hence  $f_x$  is not continuous in any neighborhood of the origin.

- (4) Viewing  $f_y$  as a function of  $y$ , it's easy to find a function  $g$  such that  $g_y = f_y$ , namely

$$g(x, y) = x^3 y^2 + x \cos(xy) + y^3$$

Now let  $f(x, y) = g(x, y) + h(x) = x^3 y^2 + x \cos(xy) + y^3 + h(x)$ . We get

$$3x^2 y^2 - xy \sin(xy) + \cos(xy) = f_x(x, y) = 3x^2 y^2 + \cos(xy) - xy \sin(xy) + h'(x)$$

Therefore  $h'(x) = 0$  and  $h(x)$  must be a constant. So  $f(x, y) = x^3 y^2 + x \cos(xy) + y^3 + C$  for some constant  $C$  will do.

(5) (2.3.21)  $D\mathbf{f}(x, y, z) = \begin{pmatrix} \frac{yz}{\sqrt{x^2+y^2+z^2}} & \frac{xz}{\sqrt{x^2+y^2+z^2}} & \frac{xy}{\sqrt{x^2+y^2+z^2}} \end{pmatrix}$ , therefore  $D\mathbf{f}(\mathbf{a})$  is  $\begin{pmatrix} 0 & -2 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \end{pmatrix}$

(6) (2.3.25)  $D\mathbf{f}(s, t) = \begin{pmatrix} 2s & 0 \\ t & s \\ 0 & 2t \end{pmatrix}$ , therefore  $D\mathbf{f}(\mathbf{a})$  is  $\begin{pmatrix} -2 & 0 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$

- (7) (2.3.30) The plane is perpendicular to the gradient vector of  $f(x, y, z) = z - 4 \cos(xy)$  at  $(\frac{\pi}{3}, 1, 2)$ , which is  $(2\sqrt{3}, \frac{2\sqrt{3}\pi}{3}, 1)$ , and passes through the point  $(\frac{\pi}{3}, 1, 2)$ . Hence the equation of the plane is  $(2\sqrt{3}, \frac{2\sqrt{3}\pi}{3}, 1) \cdot (x - \frac{\pi}{3}, y - 1, z - 2) = 0$ . Rewriting it, we get  $2\sqrt{3}x + \frac{2\sqrt{3}\pi}{3}y + z = 2 + \frac{4\sqrt{3}\pi}{3}$ .

- (8) (2.3.31) The plane is perpendicular to the gradient vector of  $f(x, y, z) = z - \exp x + y \cos(xy)$  at  $(0, 1, e)$ , which is  $(-e, -e, 1)$ , and passes through the point  $(0, 1, e)$ . Hence the equation of the plane is  $(-e, -e, 1) \cdot (x, y - 1, z - e) = 0$ . Rewriting it, we get  $ex + ey - z = 0$ .

- (9) (2.3.33)  $x_5 = -8 + (-2) \times 2(x_1 - 2) + (-6) \times (-1)(x_2 + 1) + (-4) \times 1(x_3 - 1) + (-2) \times 3(x_4 - 3) = -4x_1 + 6x_2 - 4x_3 - 6x_4 + 28$ .

(10) (2.3.51) Let  $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ . Then  $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}$ . Hence,

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = A$$

This agrees with the fact that the derivative of  $f(x) = ax$  is the slope  $a$ , when  $A$  is  $1 \times 1$  matrix.

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