S. 18.02 SOLUTIONS TO EXERCISES

1. Vectors and Matrices

1A. Vectors

1A-1 a) $|\mathbf{A}| = \sqrt{3}$, dir $\mathbf{A} = \mathbf{A}/\sqrt{3}$ b) $|\mathbf{A}| = 3$, dir $\mathbf{A} = \mathbf{A}/3$
c) $|\mathbf{A}| = 7$, dir $\mathbf{A} = \mathbf{A}/7$

1A-2 $1/25 + 1/25 + c^2 = 1 \Rightarrow c = \pm\sqrt{23}/5$

1A-3 a) $\mathbf{A} = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $|\mathbf{A}| = 3$, dir $\mathbf{A} = \mathbf{A}/3$.

b) $\mathbf{A} = |\mathbf{A}|$ dir $\mathbf{A} = 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$. Let P be its tail and Q its head. Then $OQ = OP + \mathbf{A} = 4\mathbf{j} - 3\mathbf{k}$; therefore $Q = (0, 4, -3)$.

1A-4 a) $OX = OP + PX = OP + \frac{1}{2}(PQ) = OP + \frac{1}{2}(OQ - OP) = \frac{1}{2}(OP + OQ)$
b) $OX = s\,OP + r\,OQ$; replace $\frac{1}{2}$ by r in above; use $1 - r = s$.

1A-5 $\mathbf{A} = \frac{3}{2}\sqrt{3}\,\mathbf{i} + \frac{3}{2}\mathbf{j}$. The condition is not redundant since there are two vectors of length 3 making an angle of 30^o with $\mathbf{i}$.

1A-6 wind $\mathbf{w} = 50(-\mathbf{i} - \mathbf{j})/\sqrt{2})$, $\mathbf{v} + \mathbf{w} = 200\mathbf{j} \Rightarrow \mathbf{v} = 50/\sqrt{2}\,\mathbf{i} + (200 + 50/\sqrt{2})\,\mathbf{j}$.

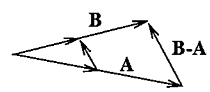
1A-7 a) $b\,\mathbf{i} - a\,\mathbf{j}$ b) $-b\,\mathbf{i} + a\,\mathbf{j}$ c) $(3/5)^2 + (4/5)^2 = 1$; $\mathbf{j}' = -(4/5)\,\mathbf{i} + (3/5)\,\mathbf{j}$

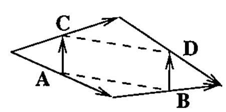
1A-8 a) is elementary trigonometry;

b) $\cos\alpha = a/\sqrt{a^2 + b^2 + c^2}$, etc.; dir $\mathbf{A} = (-1/3, 2/3, 2/3)$

c) if t, u, v are direction cosines of some $\mathbf{A}$, then $t\,\mathbf{i} + u\,\mathbf{j} + v\,\mathbf{k} =$ dir $\mathbf{A}$, a unit vector, so $t^2 + u^2 + v^2 = 1$; conversely, if this relation holds, then $t\,\mathbf{i} + u\,\mathbf{j} + v\,\mathbf{k} = \mathbf{u}$ is a unit vector, so dir $\mathbf{u} = \mathbf{u}$ and t, u, v are the direction cosines of $\mathbf{u}$.

1A-9 Letting $\mathbf{A}$ and $\mathbf{B}$ be the two sides, the third side is $\mathbf{B} - \mathbf{A}$; the line joining the two midpoints is $\frac{1}{2}\mathbf{B} - \frac{1}{2}\mathbf{A}$, which $= \frac{1}{2}(\mathbf{B} - \mathbf{A})$, a vector parallel to the third side and half its length.





1A-10 Letting $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be the four sides; then if the vectors are suitably oriented, we have $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$.

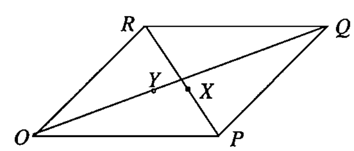
The vector from the midpoint of $\mathbf{A}$ to the midpoint of $\mathbf{C}$ is $\frac{1}{2}\mathbf{C} - \frac{1}{2}\mathbf{A}$; similarly, the vector joining the midpoints of the other two sides is $\frac{1}{2}\mathbf{B} - \frac{1}{2}\mathbf{D}$, and

$$\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D} \quad \Rightarrow \quad \mathbf{C} - \mathbf{A} = \mathbf{B} - \mathbf{D} \quad \Rightarrow \quad \tfrac{1}{2}(\mathbf{C} - \mathbf{A}) = \tfrac{1}{2}(\mathbf{B} - \mathbf{D})\ ;$$

thus two opposite sides are equal and parallel, which shows the figure is a parallelogram.

1A-11 Letting the four vertices be O, P, Q, R, with X on PR and Y on OQ,

$$\begin{aligned} OX &= OP + PX = OP + \tfrac{1}{2}PR \\ &= OP + \tfrac{1}{2}(OR - OP) \\ &= \tfrac{1}{2}(OR + OP) = \tfrac{1}{2}OQ = OY; \end{aligned}$$



therefore $X = Y$.

1B. Dot Product

1B-1 a) $\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \dfrac{4+2}{\sqrt{2}\cdot 6} = \dfrac{1}{\sqrt{2}}, \quad \theta = \dfrac{\pi}{4}$ b) $\cos\theta = \dfrac{3}{\sqrt{6}\cdot\sqrt{6}} = \dfrac{1}{2}, \quad \theta = \dfrac{\pi}{3}.$

1B-2 $\mathbf{A}\cdot\mathbf{B} = c-4$; therefore (a) orthogonal if $c=4$,

b) $\cos\theta = \dfrac{c-4}{\sqrt{c^2+5}\sqrt{6}}$; the angle θ is acute if $\cos\theta > 0$, i.e., if $c > 4$.

1B-3 Place the cube in the first octant so the origin is at one corner P, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are three edges. The longest diagonal $PQ = \mathbf{i}+\mathbf{j}+\mathbf{k}$; a face diagonal $PR = \mathbf{i}+\mathbf{j}$.

a) $\cos\theta = \dfrac{PQ\cdot PR}{|PQ|\cdot|PR|} = \dfrac{2}{\sqrt{3}\sqrt{2}}; \quad \theta = \cos^{-1}\sqrt{2/3}$

b) $\cos\theta = \dfrac{PQ\cdot\mathbf{i}}{|PQ||\mathbf{i}|} = \dfrac{1}{\sqrt{3}}, \quad \theta = \cos^{-1} 1/\sqrt{3}.$

1B-4 $QP = (a,0,-2), \quad QR = (a,-2,2)$, therefore

a) $QP\cdot QR = a^2-4$; therefore PQR is a right angle if $a^2-4=0$, i.e., if $a=\pm 2$.

b) $\cos\theta = \dfrac{a^2-4}{\sqrt{a^2+4}\sqrt{a^2+8}}$; the angle is acute if $\cos\theta > 0$, i.e., if $a^2-4>0$, or $|a|>2$, i.e., $a>2$ or $a<-2$.

1B-5 a) $\mathbf{F}\cdot\mathbf{u} = -1/\sqrt{3}$ b) $\mathbf{u} = \text{dir}\,\mathbf{A} = \mathbf{A}/7$, so $\mathbf{F}\cdot\mathbf{u} = -4/7$

1B-6 After dividing by $|OP|$, the equation says $\cos\theta = c$, where θ is the angle between OP and $\mathbf{u}$; call its solution $\theta_0 = \cos^{-1} c$. Then the locus is the nappe of a right circular cone with axis in the direction $\mathbf{u}$ and vertex angle $2\theta_0$. In particular this cone is

a) a plane if $\theta_0 = \pi/2$, i.e., if $c=0$ b) a ray if $\theta_0 = 0, \ \pi$, i.e., if $c=\pm 1$

c) nonexistent if θ_0 is nonexistent, i.e., $c>1$ or $c<-1$.

1B-7 a) $|\mathbf{i}'| = |\mathbf{j}'| = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$; a picture shows the system is right-handed.

b) $\mathbf{A}\cdot\mathbf{i}' = -1/\sqrt{2}; \qquad \mathbf{A}\cdot\mathbf{j}' = -5/\sqrt{2}$;

since they are perpendicular unit vectors, $\mathbf{A} = \dfrac{-\mathbf{i}'-5\mathbf{j}'}{\sqrt{2}}$.

c) Solving, $\mathbf{i} = \dfrac{\mathbf{i}'-\mathbf{j}'}{\sqrt{2}}, \quad \mathbf{j} = \dfrac{\mathbf{i}'+\mathbf{j}'}{\sqrt{2}}$;

thus $\mathbf{A} = 2\mathbf{i}-3\mathbf{j} = \dfrac{2(\mathbf{i}'-\mathbf{j}')}{\sqrt{2}} - \dfrac{3(\mathbf{i}'+\mathbf{j}')}{\sqrt{2}} = \dfrac{-\mathbf{i}'-5\mathbf{j}'}{\sqrt{2}}$, as before.

1B-8 a) Check that each has length 1, and the three dot products $\mathbf{i}'\cdot\mathbf{j}'$, $\mathbf{i}'\cdot\mathbf{k}'$, $\mathbf{j}'\cdot\mathbf{k}'$ are 0; make a sketch to check right-handedness.

b) $\mathbf{A}\cdot\mathbf{i}' = \sqrt{3}, \quad \mathbf{A}\cdot\mathbf{j}' = 0, \quad \mathbf{A}\cdot\mathbf{k}' = \sqrt{6}$, therefore, $\mathbf{A} = \sqrt{3}\,\mathbf{i}' + \sqrt{6}\,\mathbf{k}'$.

1B-9 Let $\mathbf{u} = \text{dir}\,\mathbf{A}$, then the vector $\mathbf{u}$-component of $\mathbf{B}$ is $(\mathbf{B}\cdot\mathbf{u})\mathbf{u}$. Subtracting it off gives a vector perpendicular to $\mathbf{u}$ (and therefore also to $\mathbf{A}$); thus

$$\mathbf{B} = (\mathbf{B}\cdot\mathbf{u})\mathbf{u} + \big(\mathbf{B} - (\mathbf{B}\cdot\mathbf{u})\mathbf{u}\big)$$

or in terms of A, remembering that $|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A}$,

$$\mathbf{B} = \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\,\mathbf{A} + \left(\mathbf{B} - \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\,\mathbf{A}\right)$$

1B-10 Let two adjacent edges of the parallelogram be the vectors $\mathbf{A}$ and $\mathbf{B}$; then the two diagonals are $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$. Remembering that for any vector $\mathbf{C}$ we have $\mathbf{C}\cdot\mathbf{C} = |\mathbf{C}|^2$, the two diagonals have equal lengths

$$\begin{aligned}
\Leftrightarrow \qquad (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) &= (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})\\
\Leftrightarrow \quad (\mathbf{A}\cdot\mathbf{A}) + 2(\mathbf{A}\cdot\mathbf{B}) + (\mathbf{B}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\mathbf{A}) - 2(\mathbf{A}\cdot\mathbf{B}) + (\mathbf{B}\cdot\mathbf{B})\\
\Leftrightarrow \qquad \mathbf{A}\cdot\mathbf{B} &= 0,
\end{aligned}$$

which says the two sides are perpendicular, i.e., the parallelogram is a rectangle.

1B-11 Using the notation of the previous exercise, we have successively,

$$\begin{aligned}
(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) &= \mathbf{A}\cdot\mathbf{A} - \mathbf{B}\cdot\mathbf{B}; \quad \text{therefore,}\\
(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = 0 \;&\Leftrightarrow\; \mathbf{A}\cdot\mathbf{A} = \mathbf{B}\cdot\mathbf{B},
\end{aligned}$$

i.e., the diagonals are perpendicular if and only if two adjacent edges have equal length, in other words, if the parallelogram is a rhombus.

1B-12 Let O be the center of the semicircle, Q and R the two ends of the diameter, and P the vertex of the inscribed angle; set $\mathbf{A} = QO = OR$ and $\mathbf{B} = OP$; then $|\mathbf{A}| = |\mathbf{B}|$.

The angle sides are $QP = \mathbf{A}+\mathbf{B}$ and $PR = \mathbf{A}-\mathbf{B}$; they are perpendicular since

$$\begin{aligned}
(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) &= \mathbf{A}\cdot\mathbf{A} - \mathbf{B}\cdot\mathbf{B}\\
&= 0, \qquad \text{since } |\mathbf{A}| = |\mathbf{B}|.
\end{aligned}$$

1B-13 The unit vectors are $\mathbf{u}_i = \cos\theta_i\,\mathbf{i} + \sin\theta_i\,\mathbf{j}$, for $i = 1, 2$; the angle between them is $\theta_2 - \theta_1$. We then have by the geometric definition of the dot product

$$\begin{aligned}
\cos(\theta_2 - \theta_1) &= \frac{\mathbf{u}_1\cdot\mathbf{u}_2}{|\mathbf{u}_1||\mathbf{u}_2|},\\
&= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2,
\end{aligned}$$

according to the formula for evaluating the dot product in terms of components.

1B-14 Let the coterminal vectors $\mathbf{A}$ and $\mathbf{B}$ represent two sides of the triangle, and let θ be the included angle. Suitably directed, the third side is then $\mathbf{C} = \mathbf{A}-\mathbf{B}$, and

$$\begin{aligned}
|\mathbf{C}|^2 = (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) &= \mathbf{A}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B} - 2\mathbf{A}\cdot\mathbf{B}\\
&= |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos\theta,
\end{aligned}$$

by the geometric interpretation of the dot product.

1C. Determinants

1C-1 a) $\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} = -1 - 8 = -9$ b) $\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix} = -10.$

1C-2 $\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix} = 2 + 0 - 8 - (24 + 4 + 0) = -34.$

a) By the cofactors of row one: $= -1\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} + 4 \cdot \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -34$

b) By the cofactors of column one: $= -1 \cdot \begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 4 \\ -2 & -1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 4 \\ 2 & 2 \end{vmatrix} = -34.$

1C-3 a) $\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -3;$ so area of the parallelogram is 3, area of the triangle is 3/2

b) sides are $PQ = (0, -3)$, $PR = (1, 1)$, $\begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix} = 3$, so area of the parallelogram is 3, area of the triangle is 3/2

1C-4 $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = x_2x_3^2 + x_1x_2^2 + x_1^2x_3 - x_1^2x_2 - x_2^2x_3 - x_1x_3^2$

$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = x_1^2x_2 - x_1^2x_3 - x_1x_3x_2 + x_1x_3^2 - x_2^2x_1 + x_2x_1x_3 + x_2^2x_3 - x_2x_3^2.$

Two terms cancel, and the other six are the same as those above, except they have the opposite sign.

1C-5 a) $\begin{vmatrix} x_1 & y_1 \\ x_2 + ax_1 & y_2 + ay_1 \end{vmatrix} = x_1y_2 + ax_1y_1 - x_2y_1 - ay_1x_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$

b) is similar.

1C-6 Use the Laplace expansion by the cofactors of the first row.

1C-7 The heads of two vectors are on the unit circle. The area of the parallelogram they span is biggest when the vectors are perpendicular, since area $= ab\sin\theta = 1 \cdot 1 \cdot \sin\theta$, and $\sin\theta$ has its maximum when $\theta = \pi/2$.

Therefore the maximum value of $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} =$ area of unit square $= 1$.

1C-9 $PQ = (0, -1, 2)$, $PR = (0, 1, -1)$, $PS = (1, 2, 1)$;

$$\text{volume parallelepiped} = \pm \begin{vmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \pm(-1) = 1\ .$$

vol. tetrahedron $= \frac{1}{3}$(base)(ht.) $= \frac{1}{3} \cdot \frac{1}{2}$ (p'piped base)(ht.) $= \frac{1}{6}$(vol. p'piped) $= 1/6$.

1C-10 Thinking of them all as origin vectors, **A** lies in the plane of **B** and **C**, therefore the volume of the parallelepiped spanned by the three vectors is zero.

1D. Cross Product

1D-1 a) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & -1 & -1 \end{vmatrix} = 3\,\mathbf{i} - (-3)\,\mathbf{j} + 3\,\mathbf{k}$ b) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -3 \\ 1 & 1 & -1 \end{vmatrix} = 3\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$

1D-2 $PQ = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $PR = -3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$, so $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -3 & 1 & -2 \end{vmatrix} = -1\,\mathbf{i} + 5\mathbf{j} + 4\,\mathbf{k}$;

area of the triangle $= \frac{1}{2}|PQ \times PR| = \frac{1}{2}\sqrt{42}$.

1D-3 We get a third vector (properly oriented) perpendicular to $\mathbf{A}$ and $\mathbf{B}$ by using $\mathbf{A} \times \mathbf{B}$:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = -\,\mathbf{i} - 2\mathbf{j} + 5\,\mathbf{k}$$

Make these unit vectors: $\mathbf{i}' = \mathbf{A}/\sqrt{5}$, $\mathbf{j}' = \mathbf{B}/\sqrt{6}$, $\mathbf{k}' = (-\,\mathbf{i} - 2\mathbf{j} + 5\,\mathbf{k})/\sqrt{30}$.

1D-4 $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$; $\quad \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

1D-5 For both, use $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta$, where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

a) $\sin\theta = 1 \Rightarrow \theta = \pi/2$; the two vectors are orthogonal.

b) $|\mathbf{A}||\mathbf{B}|\sin\theta = |\mathbf{A}||\mathbf{B}|\cos\theta$, therefore $\tan\theta = 1$, so $\theta = \pi/4$ (angle between the two vectors)

1D-6 Taking the cube so P is at the origin and three coterminous edges are $\mathbf{i}, \mathbf{j}, \mathbf{k}$, the three diagonals of the faces are $\mathbf{i} + \mathbf{j}$, $\mathbf{j} + \mathbf{k}$, $\mathbf{i} + \mathbf{k}$, so

$$\text{volume of parallelepiped spanned by diagonals} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2\,.$$

1D-7 We have $PQ = (-2,1,1)$, $PR = (-1,0,1)$, $PS = (2,1,-2)$;

$$\text{volume parallelepiped} = \pm\begin{vmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \pm 1 = 1; \quad \text{volume tetrahedron} = \frac{1}{6}.$$

1D-8 One determinant has rows in the order $\mathbf{A}, \mathbf{B}, \mathbf{C}$, the other represents $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$, and therefore has its rows in the order $\mathbf{C}, \mathbf{A}, \mathbf{B}$.

To change the first determinant into the second, interchange the second and third rows, then the first and second row; each interchange multiplies the determinant by -1, according to **D-1** (see Notes D), therefore the net effect of two successive interchanges is to leave its value unchanged; thus $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.

1D-9 a) Lift the triangle up into the plane $z = 1$, so its vertices are at the three points $P_i = (x_i, y_i, 1)$, $\quad i = 1,2,3$.

volume tetrahedron $OP_1P_2P_3 = \frac{1}{3}(\text{height})(\text{base}) = \frac{1}{3}1\cdot(\text{area of triangle})$;

volume tetrahedron $OP_1P_2P_3 = \frac{1}{6}$ (volume parallelepiped spanned by the OP_i)

$= \frac{1}{6}$ (determinant);

Therefore: area of triangle $= \frac{1}{2}$ (determinant)

b) Subtracting the first row from the second, and the first row from the third does not alter the value of the determinant, by **D-4**, and gives

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix}$$
$$= \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix},$$

using the Laplace expansion by the cofactors of the last column; but this 2×2 determinant gives the area of the parallelogram spanned by the vectors representing two sides of the plane triangle, and the triangle has half this area.

1E. Lines and Planes

1E-1 a) $(x-2)+2y-2(z+1)=0 \quad \Rightarrow \quad x+2y-2z=4.$

b) $\mathbf{N} = (1,1,0) \times (2,-1,3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & -1 & 3 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$

Removing the common factor 3, the equation is $x - y - z = 0$.

c) Calling the points respectrively P, Q, R, we have $PQ = (1,-1,1)$, $PR = (-2,3,1)$;

$\mathbf{N} = PQ \times PR = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ -2 & 3 & 1 \end{vmatrix} = -4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Equation (through $P:(1,0,1)$): $-4(x-1) - 3y + (z-1) = 0$, or $-4x - 3y + z = -3$.

d) $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1.$

e) $\mathbf{N}$ must be perpendicular to both $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $PQ = (-1,1,0)$. Therefore $\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j}$; a plane through $(1,0,1)$ with this $\mathbf{N}$ is then $x+y=1$.

1E-2 The dihedral angle between two planes is the same as the angle θ between their normal vectors. The normal vectors to the planes are respectively $(2,-1,1)$ and $(1,1,2)$; therefore $\cos\theta = \dfrac{3}{\sqrt{6}\sqrt{6}} = \dfrac{1}{2}$, so that $\theta = 60^o$ or $\pi/3$.

1E-3 a) $x = 1+2t,\ y = -t,\ z = -1+3t.$

b) $x = 2+t,\ y = -1-t,\ z = -1+2t,$ since the line has the direction of the normal to the plane.

c) The direction vector of the line should be parallel to the plane, i.e., perpendicular to its normal vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$; so the answer is

$x = 1+at,\ y = 1+bt,\ z = 1+ct,$ where $a + 2b - c = 0$, or better,

$x = 1+at,\ y = 1+bt,\ z = 1+(a+2b)t$ for any constants a, b.

1E-4 The line has direction vector $PQ = (2, -1, 1)$, so its parametric equations are:

$$x = 2t, \; y = 1 - t, \; z = 2 + t.$$

Substitute these into the equation of the plane to find a point that lies in both the line and the plane:

$$2t + 4(1 - t) + (2 + t) = 4, \quad \text{or} \quad -t + 6 = 4;$$

therefore $t = 2$, and the point is (substituting into the parametric equations): $(4, -1, 4)$.

1E-5 The line has the direction of the normal to the plane, so its parametric equations are

$$x = 1 + t, \; y = 1 + 2t, \; z = -1 - t;$$

substituting, it intersects the plane when

$$2(1 + t) - (1 + 2t) + (-1 - t) = 1, \quad \text{or} \quad -t = 1;$$

therefore, at $(0, 1, 0)$.

1E-6 Let $P_0 : (x_0, y_0, z_0)$ be a point on the plane, and $\mathbf{N} = (a, b, c)$ be a normal vector to the plane. The distance we want is the length of that origin vector which is perpendicular to the plane; but this is exactly the component of OP_0 in the direction of $\mathbf{N}$. So we get (choose the sign which makes it positive):

$$\text{distance from point to plane} = \pm\, OP_0 \cdot \frac{\mathbf{N}}{|\mathbf{N}|} = \pm\, (x_0, y_0, z_0) \cdot \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_0 + by_0 + cz_0|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}};$$

the last equality holds since the point satisfies the equation of the plane.

1F. Matrix Algebra

1F-3 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & cb + d^2 \end{pmatrix}$

We want all four entries of the product to be zero; this gives the equations:

$a^2 = -bc, \qquad b(a + d) = 0, \qquad c(a + d) = 0, \qquad d^2 = -bc.$

case 1: $a + d \neq 0$; then $b = 0$ and $c = 0$; thus $a = 0$ and $d = 0$.

case 2: $a + d = 0$; then $d = -a, \; bc = -a^2$

Answer: $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where $bc = -a^2$, i.e., $\begin{vmatrix} a & b \\ c & -a \end{vmatrix} = 0$.

1F-5 a) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$; $\quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; $\quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

Guess: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$; $\qquad$ Proof by induction:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}.$$

1F-8 a) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ d \\ g \end{pmatrix}$; $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ e \\ h \end{pmatrix}$; etc.

Answer: $\begin{pmatrix} 2 & -1 & 1 \\ 3 & 0 & 1 \\ 1 & 4 & -1 \end{pmatrix}$.

1F-9 For the entries of the product matrix $A \cdot A^T = C$, we have

$$c_{ij} = \begin{cases} 0 & \text{if } i \neq j; \\ 1 & \text{if } i = j, \end{cases} \qquad \text{since } A \cdot A^T = I.$$

On the other hand, by the definition of matrix multiplication,

$$c_{ij} = (i\text{-th row of } A)\cdot(j\text{-th column of } A^T) = (i\text{-th row of } A)\cdot(j\text{-th row of } A).$$

Since the right-hand sides of the two expressions for c_{ij} must be equal, when $j = i$ it shows that the i-th row has length 1; while for $j \neq i$, it shows that different rows are orthogonal to each other.

1G. Solving Square Systems; Inverse Matrices

1G-3 $M = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 3 & 2 \\ -2 & -1 & 1 \end{pmatrix}$; $M^T = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$; $A^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$,

where M is the matrix of cofactors (watch the signs), M^T is its transpose (the adjoint matrix), and we calculated that $\det A = 5$, to get A^{-1}. Thus

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{5}\begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 0 \\ -5 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

1G-4 The system is $A\mathbf{x} = \mathbf{y}$; the solution is $\mathbf{x} = A^{-1}\mathbf{y}$, or $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$;

written out, this is the system of equations

$$x_1 = \tfrac{3}{5}y_1 + \tfrac{1}{5}y_2 - \tfrac{2}{5}y_3, \qquad x_2 = -\tfrac{1}{5}y_1 + \tfrac{3}{5}y_2 - \tfrac{1}{5}y_3, \qquad x_3 = \tfrac{1}{5}y_1 + \tfrac{2}{5}y_2 + \tfrac{1}{5}y_3\,.$$

1G-5 Using in turn the associative law, definition of the inverse, and identity law,

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Similarly, $(AB)(B^{-1}A^{-1}) = I$. Therefore, $B^{-1}A^{-1}$ is the inverse to AB.

1H. Cramer's Rule; Theorems about Square Systems

1H-1 b) $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 2;\qquad x = \dfrac{1}{|A|}\begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 1 \end{vmatrix} = \dfrac{1}{2}\cdot 4 = 2.$

1H-2 Using Cramer's rule, the determinants in the numerators for x, y, and z all have a column of zeros, therefore have the value zero, by the determinant law **D-2**.

1H-3 a) The condition for it to have a non-zero solution is $\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -1 & c & 2 \end{vmatrix} = 0$; expanding,

$2 + 2c + 1 - (-1 + c - 4) = 0$, or $c = -8$.

b) $\begin{cases} (2-c)x + y = 0 \\ (-1-c)y = 0 \end{cases}$ has a nontrivial solution if $\begin{vmatrix} 2-c & 1 \\ 0 & -1-c \end{vmatrix} = 0$, i.e., if $(2-c)(-1-c) = 0$, or $c = 2$, $c = -1$.

c) Take $c = -8$. The equations say we want a vector (x_1, x_2, x_3) which is orthogonal to the three vectors

$$(1,-1,2),\quad (2,1,1),\quad (-1,-8,2)\ .$$

A vector orthogonal to the first two is $(1,-1,1) \times (2,1,1) = (-2,1,3)$ (by calculation). And this is orthogonal to $(-1,-8,2)$ also: $(-2,1,3)\cdot(-1,-8,2) = 0$.

1H-5 If (x_0, y_0) is a solution, then $\begin{cases} a_1x_0 + b_1y_0 = c_1 \\ a_2x_0 + b_2y_0 = c_2 \end{cases}$.

Eliminating x_0 gives $(a_2b_1 - a_1b_2)y_0 = a_2c_1 - a_1c_2$.

The left side is zero by hypothesis, so the right side is also zero: $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = 0$.

Conversely, if this holds, then a solution is $x_0 = \dfrac{c_1}{a_1}$, $y_0 = 0$ (or $x_0 = \dfrac{c_2}{a_2}$, if $a_1 = 0$).

1H-7 a) $\begin{cases} a\cos x_1 + b\sin x_1 = y_1 \\ a\cos x_2 + b\sin x_2 = y_2; \end{cases}$ has a unique solution if $\begin{vmatrix} \cos x_1 & \sin x_1 \\ \cos x_2 & \sin x_2 \end{vmatrix} \neq 0$, i.e.,

if $\cos x_1 \sin x_2 - \cos x_2 \sin x_1 \neq 0$, or equivalently, $\sin(x_2 - x_1) \neq 0$, and this last holds if and only if $x_2 - x_1 \neq n\pi$, for any integer n.

b) Since $\cos(x + n\pi) = (-1)^n \cos x$ and $\sin(x + n\pi) = (-1)^n \sin x$, the equations are solvable if and only if $y_2 = (-1)^n y_1$.

1I. Vector Functions and Parametric Equations

1I-1 Let $\mathbf{u} = \text{dir}\,(a\,\mathbf{i} + b\,\mathbf{j}) = \dfrac{a\,\mathbf{i} + b\,\mathbf{j}}{\sqrt{a^2 + b^2}}$. Then

$$\mathbf{r}(t) = \mathbf{x}_0 + vt\,\mathbf{u} = \frac{(x_0 + avt)\,\mathbf{i} + (y_0 + bvt)\,\mathbf{j}}{\sqrt{a^2+b^2}}.$$

1I-2 a) Since the motion is the reflection in the x-axis of the usual counterclockwise motion, $\mathbf{r} = a\,\cos(\omega t)\,\mathbf{i} - a\,\sin(\omega t)\,\mathbf{j}$. (This is a little special; part (b) illustrates an approach more generally applicable.)

b) The position vector is $\mathbf{r} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j}$. At time $t = 0$, the angle $\theta = \pi/2$; then it decreases linearly at the rate ω. Therefore $\theta = \pi/2 - \omega t$; substituting and then using the trigonometric identities for $\cos(A+B)$ and $\sin(A+B)$, we get

$$\mathbf{r} = a\cos(\pi/2 - \omega t)\,\mathbf{i} + a\sin(\pi/2 - \omega t)\,\mathbf{j} = a\sin\omega t\,\mathbf{i} + a\cos\omega t\,\mathbf{j}$$

(In retrospect, we could have given another "special" derivation by observing that this motion is the reflection in the diagonal line $y = x$ of the usual counterclockwise motion

(In retrospect, we could have given another "special" derivation by observing that this motion is the reflection in the diagonal line $y = x$ of the usual counterclockwise motion starting at $(a, 0)$, so we get its position vector $\mathbf{r}(t)$ by interchanging the x and y in the usual position vector function $\mathbf{r} = \cos(\omega t)\,\mathbf{i} + \sin(\omega t)\,\mathbf{j}$.)

1I-3 a) $x = 2\cos^2 t,\ y = \sin^2 t,$ so $x + 2y = 2$; only the part of this line between $(0,1)$ and $(2,0)$ is traced out, back and forth.

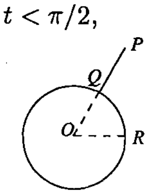
b) $x = \cos 2t,\ y = \cos t$; eliminating t to get the xy-equation, we have

$$\cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 \quad \Rightarrow \quad x = 2y^2 - 1\ ;$$

only the part of this parabola between $(1,1)$ and $(1,-1)$ is traced out, back and forth.

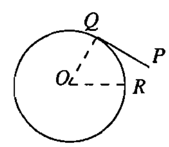
c) $x = t^2 + 1,\ y = t^3$; eliminating t, we get $y^2 = (x-1)^3$; the entire curve is traced out as t increases, with y going from $-\infty$ to ∞.

d) $x = \tan t,\ y = \sec t$; eliminate t via the trigonometric identity $\tan^2 t + 1 = \sec^2 t$, getting $y^2 - x^2 = 1$. This is a hyperbola; the upper branch is traced out for $-\pi/2 < t < \pi/2$, the lower branch for $\pi/2 < t < 3\pi/2$. Then it repeats.



1I-4 $OP = |OP| \cdot \text{dir}\, OP$; $\text{dir}\, OP = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$;
$|OP| = |OQ| + |QP| = a + a\theta$, since $|QP| = \text{arc}\, QR = a\theta$.

So $OP = a(1+\theta)(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$ or $x = a(1+\theta)\cos\theta,\ y = a(1+\theta)\sin\theta$.



1I-5 $OP = OQ + QP$; $OQ = a(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$
$QP = |QP|\ \text{dir}\, QP = a\theta(\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j})$, since $|QP| = \text{arc}\, QR = a\theta$
(cf. Exer. 1A-7a for dir QP)

Therefore, $OP = \mathbf{r} = a(\cos\theta + \theta\sin\theta)\,\mathbf{i} + a(\sin\theta - \theta\cos\theta)\,\mathbf{j}$

1I-6 a) $\mathbf{r}_1(t) = (10 - t)\,\mathbf{i}$ (hunter); $\mathbf{r}_2(t) = t\,\mathbf{i} + 2t\,\mathbf{j}$ (rabbit; note that $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, so indeed $|\mathbf{v}| = \sqrt{5}$)

$$\begin{aligned} \text{Arrow} = HA &= \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}, \text{ since the arrow has unit length} \\ &= \frac{(t-5)\,\mathbf{i} + t\,\mathbf{j}}{\sqrt{2t^2 - 10t + 25}}, \text{ after some algebra.} \end{aligned}$$

b) It is easier mathematically to minimize the square of the distance between hunter and rabbit, rather than the distance itself; you get the same t-value in either case.

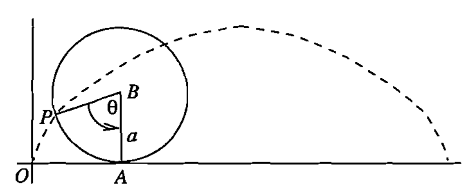
Let $f(t) = |\mathbf{r}_2 - \mathbf{r}_1|^2 = 2t^2 - 10t + 25$; then $f'(t) = 4t - 10 = 0$ when $t = 2.5$.

1I-7 $OP = OA + AB + BP$;

$OA = \text{arc}AP = a\theta$; $AB = a\,\mathbf{j}$;

$BP = a(-\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j})$

Therefore, $OP = a(\theta - \sin\theta)\,\mathbf{i} + a(1 - \cos\theta)\,\mathbf{j}$.



1J. Vector Derivatives

1J-1 a) $\mathbf{r} = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$; $\quad \mathbf{v} = e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}$, $\quad |\mathbf{v}| = \sqrt{e^{2t} + e^{-2t}}$, $\quad \mathbf{T} = \dfrac{e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$,
$\mathbf{a} = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$

b) $\mathbf{r} = t^2\,\mathbf{i} + t^3\,\mathbf{j}$; $\quad \mathbf{v} = 2t\,\mathbf{i} + 3t^2\,\mathbf{j}$; $\quad |\mathbf{v}| = t\sqrt{4 + 9t^2}$; $\quad \mathbf{T} = \dfrac{2\,\mathbf{i} + 3t\,\mathbf{j}}{\sqrt{4 + 9t^2}}$;
$\mathbf{a} = 2\,\mathbf{i} + 6t\,\mathbf{j}$

c) $\mathbf{r} = (1 - 2t^2)\,\mathbf{i} + t^2\,\mathbf{j} + (-2 + 2t^2)\,\mathbf{k}$; $\quad \mathbf{v} = 2t(-2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k})$; $\quad |\mathbf{v}| = 6t$;
$\mathbf{T} = \frac{1}{3}(-2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k})$; $\quad \mathbf{a} = 2(-2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k})$

1J-2 a) $\mathbf{r} = \dfrac{1}{1+t^2}\,\mathbf{i} + \dfrac{t}{1+t^2}\,\mathbf{j}$; $\quad \mathbf{v} = \dfrac{-2t\,\mathbf{i} + (1-t^2)\,\mathbf{j}}{(1+t^2)^2}$; $\quad |\mathbf{v}| = \dfrac{1}{1+t^2}$; $\quad \mathbf{T} = \dfrac{-2t\,\mathbf{i} + (1-t^2)\,\mathbf{j}}{1+t^2}$

b) $|\mathbf{v}|$ is largest when $t = 0$, therefore at the point $(1, 0)$. There is no point at which $|\mathbf{v}|$ is smallest; as $t \to \infty$ or $t \to -\infty$, the point $P \to (0,0)$, and $|\mathbf{v}| \to 0$.

c) The position vector shows $y = tx$, so $t = y/x$; substituting into $x = 1/(1+t^2)$ yields after some algebra the equation $x^2 + y^2 - x = 0$; completing the square gives the equation $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$, which is a circle with center at $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$.

1J-3 $$\frac{d}{dt}(x_1 y_1 + x_2 y_2) = \begin{cases} x_1' y_1 + x_1 y_1' + \\ x_2' y_2 + x_2 y_2' \end{cases}$$

Adding the columns, we get: $(x_1, x_2)' \cdot (y_1, y_2) + (x_1, x_2) \cdot (y_1, y_2)' = \dfrac{d\mathbf{r}}{dt} \cdot \mathbf{s} + \mathbf{r} \cdot \dfrac{d\mathbf{s}}{dt}$.

1J-4 a) Since P moves on a sphere, say of radius a,

$$x(t)^2 + y(t)^2 + z(t)^2 = a^2;$$

Differentiating,

$$2xx' + 2yy' + 2zz' = 0,$$

which says that $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \cdot x'\,\mathbf{i} + y'\,\mathbf{j} + z'\,\mathbf{k} = 0$ for all t, i.e., $\mathbf{r} \cdot \mathbf{r}' = 0$.

b) Since by hypothesis, $\mathbf{r}(t)$ has length a, for all t, we get the chain of implications

$$|\mathbf{r}| = a \quad \Rightarrow \quad \mathbf{r} \cdot \mathbf{r} = a^2 \quad \Rightarrow \quad 2r \cdot \frac{d\mathbf{r}}{dt} = 0 \quad \Rightarrow \quad \mathbf{r} \cdot \mathbf{v} = 0.$$

c) Using first the result in Exercise **1J-3**, then $\mathbf{r} \cdot \mathbf{r} = |\mathbf{r}|^2$, we have

$$\mathbf{r} \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \frac{d}{dt}\mathbf{r} \cdot \mathbf{r} = 0 \quad \Rightarrow \quad \mathbf{r} \cdot \mathbf{r} = c, \text{ a constant}, \quad \Rightarrow \quad |\mathbf{r}| = \sqrt{c},$$

which shows that the head of $\mathbf{r}$ moves on a sphere of radius $\sqrt{c}$.

1J-5 a) $|\mathbf{v}| = c \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{v} = c^2 \quad \Rightarrow \quad \dfrac{d}{dt}\mathbf{v} \cdot \mathbf{v} = 2\mathbf{v} \cdot \mathbf{a} = 0$, by **1J-3**. Therefore the velocity and acceleration vectors are perpendicular.

b) $\mathbf{v} \cdot \mathbf{a} = 0 \quad \Rightarrow \quad \frac{d}{dt}\mathbf{v} \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{v} = a \quad \Rightarrow \quad |\mathbf{v}| = \sqrt{a}$, which shows the speed is constant.

1J-6 a) $\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$; $\quad \mathbf{v} = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k}$; $\quad |\mathbf{v}| = \sqrt{a^2 + b^2}$;
$\mathbf{T} = \dfrac{\mathbf{v}}{\sqrt{a^2 + b^2}}$; $\quad \mathbf{a} = -a(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$

b) By direct calculation using the components, we see that $\mathbf{v} \cdot \mathbf{a} = 0$; this also follows theoretically from Exercise **1J-5b**, since the speed is constant.

1J-7 a) The criterion is $|\mathbf{v}| = 1$; namely, if we measure arclength s so $s = 0$ when $t = 0$, then since s increases with t,

$$|\mathbf{v}| = 1 \quad \Rightarrow \quad ds/dt = 1 \quad \Rightarrow \quad s = t + c \quad \Rightarrow \quad s = t\ .$$

b) $\mathbf{r} = (x_0 + at)\,\mathbf{i} + (y_0 + at)\,\mathbf{j}$; $\mathbf{v} = a(\mathbf{i} + \mathbf{j})$; $|\mathbf{v}| = a\sqrt{2}$; therefore choose $a = 1/\sqrt{2}$.

c) Choose a and b to be non-negative numbers such that $a^2 + b^2 = 1$; then $\mathbf{v}| = 1$.

1J-8 a) Let $\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$; then $u(t)\mathbf{r}(t) = ux\,\mathbf{i} + uy\,\mathbf{j}$, and differentiation gives

$$(u\mathbf{r})' = (ux)'\,\mathbf{i} + (uy)'\,\mathbf{j} = (u'x + ux')\,\mathbf{i} + (u'y + uy')\,\mathbf{j} = u'(x\,\mathbf{i} + y\,\mathbf{j}) + u(x'\,\mathbf{i} + y'j) = u'\mathbf{r} + u\mathbf{r}'.$$

b)
$$\begin{aligned}\frac{d}{dt}\, e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j}) &= e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j}) + e^t(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}) \\ &= e^t\big((\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j}\big).\end{aligned}$$

Therefore $|\mathbf{v}| = e^t\big|(\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j}\big| = 2e^t$, after calculation.

1J-9 a) $\mathbf{r} = 3\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j} + 4\cos t\,\mathbf{k} \quad \Rightarrow \quad |\mathbf{r}| = \sqrt{25\cos^2 t + 25\sin^t} = 5.$

b) $\mathbf{v} = -3\sin t\,\mathbf{i} + 5\cos t\,\mathbf{j} - 4\sin t\,\mathbf{k}$; therefore $|\mathbf{v}| = \sqrt{25\cos^2 t + 25\sin^2 t} = 5.$

c) $\mathbf{a} = d\mathbf{v}/dt = -3\cos t\,\mathbf{i} - 5\sin t\,\mathbf{j} - 4\cos t\,\mathbf{k} = -\mathbf{r}$

d) By inspection, the x, y, z coordinates of p satisfy $4x - 3z = 0$, which is a plane through the origin.

e) Since by part (a) the point P moves on the surface of a sphere of radius 5 centered at the origin, and by part (d) also in a plane through the origin, its path must be the intersection of these two surfaces, which is a great circle of radius 5 on the sphere.

1J-10 a) Use the results of Exercise **1J-6**:

$$\mathbf{T} = \frac{-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k}}{\sqrt{a^2 + b^2}}; \qquad |\mathbf{v}| = \sqrt{a^2 + b^2}\ .$$

By the chain rule,

$$\left|\frac{d\mathbf{T}}{dt}\right| = \left|\frac{d\mathbf{T}}{ds}\right|\left|\frac{ds}{dt}\right|;$$

therefore

$$\frac{|-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}|}{\sqrt{a^2 + b^2}} = \kappa\sqrt{a^2 + b^2}\ ;$$

since the numerator on the left has the value $|a|$, we get

$$\kappa = \frac{|a|}{a^2 + b^2}\ .$$

b) If $b = 0$, the helix is a circle of radius $|a|$ in the xy-plane, and $\kappa = \dfrac{1}{|a|}$.

1K. Kepler's Second Law

1K-1 $\frac{d}{dt}(x_1y_1 + x_2y_2) \;=\; \begin{cases} x_1'y_1 + x_1y_1' + \\ x_2'y_2 + x_2y_2' \end{cases}$

Adding the columns, we get: $\langle x_1, x_2\rangle' \cdot \langle y_1, y_2\rangle + \langle x_1, x_2\rangle \cdot \langle y_1, y_2\rangle' \;=\; \frac{d\mathbf{r}}{dt}\cdot \mathbf{s} + \mathbf{r}\cdot\frac{d\mathbf{s}}{dt}$.

1K-2 In two dimensions, $\mathbf{s}(t) = \langle x(t), y(t)\rangle$, $\mathbf{s}'(t) = \langle x'(t), y'(t)\rangle$.

Therefore $\mathbf{s}'(t) = \mathbf{0} \;\Rightarrow\; x'(t) = 0,\; y'(t) = 0 \;\Rightarrow\; x(t) = k_1,\; y(t) = k_2 \;\Rightarrow\; \mathbf{s}(t) = (k_1, k_2)$ where k_1, k_2 are constants.

1K-3 Since $\mathbf{F}$ is central, we have $\mathbf{F} = c\mathbf{r}$; using Newton's law, $\mathbf{a} = \mathbf{F}/m \;=\; (c/m)\mathbf{r}$; so

$$
\begin{aligned}
\mathbf{F} = c\mathbf{r} \quad &\Rightarrow \quad \mathbf{r}\times\mathbf{a} \;=\; \mathbf{r}\times\frac{c}{m}\mathbf{r} \;=\; \mathbf{0}, \\
&\Rightarrow \quad \frac{d}{dt}(\mathbf{r}\times\mathbf{v}) \;=\; \mathbf{0}, \qquad \text{by (7)} \\
(*) \qquad &\Rightarrow \quad \mathbf{r}\times\mathbf{v} \;=\; \mathbf{K}, \quad \text{a constant vector, by Exercise K-2.}
\end{aligned}
$$

This last line (*) shows that $\mathbf{r}$ is perpendicular to $\mathbf{K}$, and therefore its head (the point P) lies in the plane through the origin which has $\mathbf{K}$ as normal vector. Also, since

$$
\begin{aligned}
|\mathbf{r}\times\mathbf{v}| \;&=\; 2\,\frac{dA}{dt}, \quad \text{by (2)}, \\
|\mathbf{r}\times\mathbf{v}| \;&=\; |K|, \quad \text{by } (*),
\end{aligned}
$$

we conclude that

$$
\frac{dA}{dt} \;=\; \frac{1}{2}\,|K|\,,
$$

which shows the area is swept out at a constant rate.