 OK, so remember, we've seen Stokes theorem, which says if I have a closed curve bounding some surface, S, and I orient the curve and the surface compatible with each other, then I can compute the line integral along C along my curve in terms of, instead, surface integral for flux of a different vector field, namely, curl f dot n dS. OK, so that's the statement. And, just to clarify a little bit, so, again, we've seen various kinds of integrals. So, line integrals we know how to evaluate. They take place in a curve. You express everything in terms of one variable, and after substituting, you end up with a usual one variable integral that you know how to evaluate. And, surface integrals, we know also how to evaluate. Namely, we've seen various formulas for ndS. Once you have such a formula, due to the dot product with this vector field, which is not the same as that one. But it's a new vector field that you can build out of f. You do the dot product. You express everything in terms of your two integration variables, and then you evaluate. So, now, what does this have to do with various other things? So, one thing I want to say has to do with how Stokes helps us understand path independence, so, how it actually motivates our criterion for gradient fields, independence. OK, so, we've seen that if we have a vector field defined in a simply connected region, and its curl is zero, then it's a gradient field, and the line integral is path independent. So, let me first define for you when a simply connected region is. So, we say that a region in space is simply connected -- -- if every closed loop inside this region bounds some surface again inside this region. OK, so let me just give you some examples just to clarify. So, for example, let's say that I have a region that's the entire space with the origin removed. OK, so space with the origin removed, OK, you think it's simply connected? Who thinks it's simply connected? Who thinks it's not simply connected? Let's think a little bit harder. Let's say that I take a loop like this one, OK, it doesn't go through the origin. Can I find a surface that's bounded by this loop and that does not pass through the origin? Yeah, I can take the sphere, you know, for example, or anything that's just not quite the disk? So, and similarly, if I take any other loop that avoids the origin, I can find, actually, a surface bounded by it that does not pass through the origin. So, actually, that's kind of a not so obvious theorem to prove, but maybe intuitively, start by finding any surface. Well, if that surface passes through the origin, just wiggle it a little bit, you can make sure it doesn't pass through the origin anymore. Just push it a little bit. So, in fact, this is simply connected. That was a trick question. OK, now on the other hand, a good example of something that is not simply connected is if I take space, and I remove the z axis -- -- that is not simply connected. And, see, the reason is, if I look again, say, at the unit circle in the x axis, sorry, unit circle in the xy plane, I mean, in the xy plane, so, if I try to find a surface whose boundary is this disk, well, it has to actually cross the z axis somewhere. There's no way that I can find a surface whose only boundary is this curve, which doesn't hit the z axis anywhere. Of course, you could try to use the same trick as there, say, maybe we want to go up, up, up. You know, let's start with a cylinder. Well, the problem is you have to go infinitely far because the z axis goes infinitely far. And, you'll never be able to actually close your surface. So, the matter what kind of trick you might want to use, it's actually a theorem in topology that you cannot find a surface bounded by this disk without intersecting the z axis. Yes? Well, a doughnut shape certainly would stay away from the z axis, but it wouldn't be a surface with boundary just this guy. Right, it would have to have either some other boundary. So, maybe what you have in mind is some sort of doughnut shape like this that curves on itself, and maybe comes back. Well, if you don't quite close it all the way around, so I can try to, indeed, draw some sort of doughnut here. Well, if I don't quite close it, that it will have another edge at the other end wherever I started. If I close it completely, then this curve is no longer its boundary because my surface lives on both sides of this curve. See, I want a surface that stops on this curve, and doesn't go beyond it. And, nowhere else does it have that kind of behavior. Everywhere else, it keeps going on. So, actually, I mean, maybe actually another way to convince yourself is to find a counter example to the statement I'm going to make about vector fields with curl zero and simply connected regions always being conservative. So, what you can do is you can take the example that we had in one of our older problem sets. That was a vector field in the plane. But, you can also use it to define a vector field in space just with no z component. That vector field is actually defined everywhere except on the z axis, and it violates the usual theorem that we would expect. So, that's one way to check just for sure that this thing is not simply connected. So, what's the statement I want to make? So, recall we've seen if F is a gradient field -- -- then its curl is zero. That's just the fact that the mixed second partial derivatives are equal. So, now, the converse is the following theorem. It says if the curl of F equals zero in, sorry, and F is defined -- No, is not the logical in which to say it. So, if F is defined in a simply connected region, and curl F is zero -- -- then F is a gradient field, and the line integral for F is path independent -- -- F is conservative, and so on, all the usual consequences. Remember, these are all equivalent to each other, for example, because you can use path independence to define the potential by doing the line integral of F. OK, so where do we use the assumption of being defined in a simply connected region? Well, the way which we will prove this is to use Stokes theorem. OK, so the proof, so just going to prove that the line integral is path independent; the others work the same way. OK, so let's assume that we have a vector field whose curl is zero. And, let's say that we have two curves, C1 and C2, that go from some point P0 to some point P1, the same point to the same point. Well, we'd like to understand the line integral along C1, say, minus the line integral along C2 to show that this is zero. That's what we are trying to prove. So, how will we compute that? Well, the line integral along C1 minus C2, well, let's just form a closed curve that is C1 minus C2. OK, so let's call C, woops -- So that's equal to the integral along C of f dot dr where C is C1 followed by C2 backwards. Now, C is a closed curve. So, I can use Stokes theorem. Well, to be able to use Stokes theorem, I need, actually, to find a surface to apply it to. And, that's where the assumption of simply connected is useful. I know in advance that any closed curve, so, C in particular, has to bound some surface. OK, so we can find S, a surface, S, that bounds C because the region is simply connected. So, now that tells us we can actually apply Stokes theorem, except it won't fit here. So, instead, I will do that on the next line. That's equal by Stokes to the double integral over S of curl F dot vector dS, or ndS. But now, the curl is zero. So, if I integrate zero, I will get zero. OK, so I proved that my two line integrals along C1 and C2 are equal. But for that, I needed to be able to find a surface which to apply Stokes theorem. And that required my region to be simply connected. If I had a vector field that was defined only outside of the z axis and I took two paths that went on one side and the other side of the z axis, I might have obtained, actually, different values of the line integral. OK, so anyway, that's the customary warning about simply connected things. OK, let me just mention very quickly that there's a lot of interesting topology you can do, actually in space. So, for example, this concept of being simply connected or not, and studying which loops bound surfaces or not can be used to classify shapes of things inside space. So, for example, one of the founding achievements of topology in the 19th century was to classify surfaces in space -- -- by trying to look at loops on them. So, what I mean by that is that if I take the surface of a sphere, well, I claim the surface of a sphere -- -- is simply connected. Why is that? Well, let's take my favorite closed curve on the surface of a sphere. I can always find a portion of the sphere that's bounded by it. OK, so that's the definition of the surface of a sphere being simply connected. On the other hand, if I take what's called a torus, or if you prefer, the surface of a doughnut, that's more, it's a less technical term, but it's -- -- well, that's not simply connected. And, in fact, for example, if you look at this loop here that goes around it, well, of course it bounds a surface in space. But, that surface cannot be made to be just a piece of the donut. You have to go through the hole. You have to leave the surface of a torus. In fact, there's another one. See, this one also does not bound anything that's completely contained in the torus. And, of course, it bounds this disc, but inside of a torus. But, that's not a part of the surface itself. So, in fact, there's, and topologists would say, there's two independent -- -- loops that don't bound surfaces, that don't bound anything. And, so this number two is somehow an invariant that you can associate to this kind of shape. And then, if you consider more complicated surfaces with more holes in them, you can try, somehow, to count independent loops on them, and that's the beginning of the classification of surfaces. Anyway, that's not really an 18.02 topic, but I thought I would mentioned it because it's kind of a cool idea. OK, let me say a bit more in the way of fun remarks like that. So, food for thought: let's say that I want to apply Stokes theorem to simplify a line integral along the curve here. So, this curve is maybe not easy to see in the picture. It kind of goes twice around the z axis, but spirals up and then down. OK, so one way to find a surface that's bounded by this curve is to take what's called the Mobius strip. OK, so the Mobius strip, it's a one sided strip where when you go around, you flip one side becomes the other. So, you just, if you want to take a band of paper and glue the two sides with a twist, so, it's a one sided surface. And, that gives us, actually, serious trouble if we try to orient it to apply Stokes theorem. So, see, for example, if I take this Mobius strip, and I try to find an orientation, so here it looks like that, well, let's say that I've oriented my curve going in this direction. So, I go around, around, around, still going this direction. Well, the orientation I should have for Stokes theorem is that when I, so, curve continues here. Well, if you look at the convention around here, it tells us that the normal vector should be going this way. OK, if we look at it near here, if we walk along this way, the surface is to our right . So, we should actually be flipping things upside down. The normal vector should be going down. And, in fact, if you try to follow your normal vector that's pointing up, it's pointing up, up, up. It will have to go into things, into, into, down. There's no way to choose consistently a normal vector for the Mobius strip. So, that's what we call a non-orientable surface. And, that just means it has only one side. And, if it has only one side, that we cannot speak of flux for it because we have no way of saying that we'll be counting things positively one way, negatively the other way, because there's only one, you know, there's no notion of sides. So, you can't define a side towards which things will be going positively. So, that's actually a situation where flux cannot be defined. OK, so as much as Mobius strips and climb-bottles are exciting and really cool, well, we can't use them in this class because we can't define flux through them. So, if we really wanted to apply Stokes theorem, because I've been telling you that space is simply connected, and I will always be able to apply Stokes theorem to any curve, what would I do? Well, I claim this curve actually bounds another surface that is orientable. Yeah, that looks counterintuitive. Well, let's see it. I claim you can take a hemisphere, and you can take a small thing and twist it around. So, in case you don't believe me, let me do it again with the transparency. Here's my loop, and see, well, the scale is not exactly the same. So, it doesn't quite match. But, and it's getting a bit dark. But, that spherical thing with a little slit going twisting into it will actually have boundary my loop. And, that one is orientable. I mean, I leave it up to you to stare at the picture long enough to convince yourselves that there's a well-defined up and down. OK. So now, I mean, in case you are getting really, really worried, I mean, there won't be any Mobius strips on the exam on Tuesday, OK? It's just to show you some cool stuff. OK, questions? No? OK, one last thing I want to show you before we start reviewing, so one question you might have about Stokes theorem is, how come we can choose whatever surface we want? I mean, sure, it seems to work, but why? So, I'm going to say a couple of words about surface independence in Stokes theorem. So, let's say that I have a curve, C, in space. And, let's say that I want to apply Stokes theorem. So, then I can choose my favorite surface bounded by C. So, in a situation like this, for example, I might want to make my first choice be this guy, S1, like maybe some sort of upper half sphere. And, if you pay attention to the orientation conventions, you'll see that you need to take it with normal vector pointing up. Maybe actually I would rather make a different choice. And actually, I will choose another surface, S2, that maybe looks like that. And, if I look carefully at the orientation convention, Stokes theorem tells me that I have to take the normal vector pointing up again. So, that's actually into things. So, Stokes says that the line integral along C of my favorite vector field can be computed either as a flux integral for the curl through S1, or as the same integral, but through S2 instead of S1. So, that seems to suggest that curl F has some sort of surface independence property. It doesn't really matter which surface I take, as long as the boundary is this given curve, C. Why is that? That's a strange property to have. Where does it come from? Well, let's think about it for a second. So, why are these the same? I mean, of course, they have to be the same because that's what Stokes tell us. But, why is that OK? Well, let's think about comparing the flux integral for S1 and the flux integral for S2. So, if we want to compare them, we should probably subtract them from each other. OK, so let's do the flux integral for S1 minus the flux integral for S2 of the same thing. Well, let's give a name. Let's call S the surface S1 minus S2. So, what is S? S is S1 with its given orientation together with S2 with the reversed orientation. So, S is actually this whole closed surface here. And, the normal vector to S seems to be pointing outwards everywhere. OK, so now, if we have a closed surface with a normal vector pointing outwards, and we want to find a flux integral for it, well, we can replace that with a triple integral. So, that's the divergence theorem. So, that's by the divergence theorem using the fact that S is a closed surface. That's equal to the triple integral over the region inside. Let me call that region D of divergence, of curl F dV. OK, and what I'm going to claim now is that we can actually check that if you take the divergence of the curl of a vector field, you always get zero. OK, and so that will tell you that this integral will always be zero. And that's why the flux for S1, and the flux for S2 were the same a priori and we didn't have to worry about which one we chose when we did Stokes theorem. OK, so let's just check quickly that divergence of a curve is zero. OK, in case you're wondering why I'm doing all this, well, first I think it's kind of interesting, and second, it reminds you of a statement of all these theorems, and all these definitions. So, in a way, we are already reviewing. OK, so let's see. If my vector field has components P, Q, and R, remember that the curl was defined by this cross product between del and our given vector field. So, that's Ry - Qz followed by Pz - Rx, and Qx - Py. So, now, we want to take the divergence of this. Well, so we have to take the first component, Ry minus Qz, and take its partial with respect to x. Then, take the y component, Pz minus Rx partial with respect to y plus Qx minus Py partial with respect to z. And, well, now we should expand this. But I claim it will always simplify to zero. OK, so I think we have over there, becomes R sub yx minus Q sub zx plus P sub zy minus R sub xy plus Q sub xz minus P sub yz. Well, let's see. We have P sub zy minus P sub yz. These two cancel out. We have R sub yx minus R sub xy. These cancel out. Q sub zx and Q sub xz, these two also cancel out. So, indeed, the divergence of a curl is always zero. OK, so the claim is divergence of curl is always zero. Del cross F is always zero, and just a small remark, if we had actually real vectors rather than this strange del guy, indeed we know that if we have two vectors, U and V, and we do u dot u cross v, what is that? Well, one way to say it is it's the determinant of u, u, and v, which is the volume of the box. But, it's completely flat because u, u, and v are all in the plane defined by u and v. The other way to say it is that u cross v is perpendicular to u and v. Well, if it's perpendicular u, then its dot product with u will be zero. So, no matter how you say it, this is always zero. So, in a way, this reinforces our intuition that del, even though it's not at all an actual vector sometimes can be manipulated in the same way. OK, I think that's it for new topics for today. And, so, now I should maybe try to recap quickly what we've learned in these past three weeks so that you know, so, the exam is probably going to be similar in difficulty to the practice exams. That's my goal. I don't know if I will have reached that goal or not. We'll only know that after you've taken the test. But, the idea is it's meant to be more or less the same level of difficulty. So, at this point, we've learned about three kinds of beasts in space. OK, so I'm going to divide my blackboard into three pieces, and here I will write triple integrals. We've learned about double integrals, and we've learned about line integrals. OK, so triple integrals over a region in space, we integrate a scalar quantity, dV. How do we do that? Well, we can do that in rectangular coordinates where dV becomes something like, maybe, dz dx dy, or any permutation of these. We've seen how to do it also in cylindrical coordinates where dV is maybe dz times r dr d theta or more commonly r dr d theta dz. But, what I want to emphasize in this way is that both of these you set up pretty much in the same way. So, remember, the main trick here is to find the bounds of integration. So, when you do it, say, with dz first, that means for fixed xy, so, for a fixed point in the xy plane, you have to look at the bounds for z. So, that means you have to figure out what's the bottom surface of your solid, and what's the top surface of your solid? And, you have to find the value of z at the bottom, the value of z at the top as functions of x and y. And then, you will put that as bounds for z. Once you've done that, you are left with the question of finding bounds for x and y. Well, for that, you just rotate the picture, look at your solid from above, so, look at its projection to the xy plane, and you set up a double integral either in rectangular xy coordinates, or in polar coordinates for x and y. Of course, you can always do it a different orders. And, I'll let you figure out again how that goes. But, if you do dz first, then the inner bounds are given by bottom and top, and the outer ones are given by looking at the shadow of the region. Now, there's also spherical coordinates. And there, we've seen that dV is rho squared sine phi d rho d phi d theta. So now, of course, if this orgy of Greek letters is confusing you at this point, then you probably need to first review spherical coordinates for themselves. Remember that rho is the distance from the origin. Phi is the angle down from the z axis. So, it's zero, and the positive z axis, pi over two in the xy plane, and increases all the way to pi on the negative z axis. And, theta is the angle around the z axis. So, now, when we set up bounds here, it will look a lot like what you've done in polar coordinates in the plane because when you look at the inner bound down on rho, for a fixed phi and theta, that means you're shooting a straight ray from the origin in some direction in space. So, you know, you're sending a laser beam, and you want to know what part of your beam is going to be in your given solid. You want to solve for the value of rho when you enter the solid and when you leave it. I mean, very often, if the origin is in your solid, then rho will start at zero. Then you want to know when you exit. And, I mean, there's a fairly small list of kinds of surfaces that we've seen how to set up in spherical coordinates. So, if you're really upset by this, go over the problems in the notes. That will give you a good idea of what kinds of things we've seen in spherical coordinates. OK, and then evaluation is the usual way. Questions about this? No? OK, so, I should say we can do something bad, but so we've seen, of course, applications of this. So, we should know how to use a triple integral to evaluate things like a mass of a solid, the average value of a function, the moment of inertia about one of the coordinate axes, or the gravitational attraction on a mass at the origin. OK, so these are just formulas to remember for examples of triple integrals. It doesn't change conceptually. You always set them up and evaluate them the same way. It just tells you what to put there for the integrand. Now, double integrals: so, when we have a surface in space, well, what we will integrate on it, at least what we've seen how to integrate is a vector field dotted with the unit normal vector times the area element. OK, and this is sometimes called vector dS. Now, how do we evaluate that? Well, we've seen formulas for ndS in various settings. And, once you have a formula for ndS, that will relate ndS to maybe dx dy, or something else. And then, you will express, so, for example, ndS equals something dx dy. And then, it becomes a double integral of something dx dy. Now, in the integrand, you want to express everything in terms of x and y. So, if you had a z, maybe you have a formula for z in terms of x and y. And, when you set up the bounds, well, you try to figure out what are the bounds for x and y? That would be just looking at it from above. Of course, if you are using other variables, figure out the bounds for those variables. And, when you've done that, it becomes just a double integral in the usual sense. OK, so maybe I should be a bit more explicit about formulas because there have been a lot. So, let me tell you about a few of them. Let me actually do that over here because I don't want to make this too crowded. OK, so what kinds of formulas for ndS have we seen? Well, we've seen a formula, for example, for a horizontal plane, or for something that's parallel to the yz plane or the xz plane. Well, let's do just the yz plane for a quick reminder. So, if I have a surface that's contained inside the yz plane, then obviously I will express ds in terms of, well, I will use y and z as my variables. So, I will say that ds is dy dz, or dz dy, whatever's most convenient. Maybe we will even switch to polar coordinates after that if a problem wants us to. And, what about the normal vector? Well, the normal vector is either coming straight at us, or it's maybe going back away from us depending on which orientation we've chosen. So, this gives us ndS. We dot our favorite vector field with it. We integrate, and we get the answer. OK, we've seen about spheres and cylinders centered at the origin or centered on the z axis. So, the normal vector sticks straight out or straight in, depending on which direction you do it in. So, for a sphere, the normal vector is divided by the radius of the sphere. For a cylinder, it's , divided by the radius of a cylinder. And, the surface element on a sphere, so, see, it's very closely related to the volume element of spherical coordinates except you don't have a rho anymore. You just plug in a rho equals a. So, you get a squared sine phi d phi d theta. And, for a cylinder, it would be a dz d theta. So, by the way, just a quick check, when you're doing an integral, if it's the surface integral, there should be two integral signs, and there should be two integration variables. And, there should be two d somethings. If you end up with a dx, dy, dz in the surface integral, something is seriously wrong. OK, now, besides these specific formulas, we've seen two general formulas that are also useful. So, one is, if we know how to express z in terms of x and y, and just to change notation to show you that it's not set in stone, let's say that z is known as a function z of x and y. So, how do I get ndS in that case? Well, we've seen a formula that says negative partial z partial x, negative partial z partial y, one dx dy. So, this formula relates the volume, sorry, the surface element on our surface to the area element in the xy plane. It lets us convert between dS and dx dy. OK, so we just plug in this, and we dot with F, and then we substitute everything in terms of x and y, and we evaluate the integral over x and y. If we don't really want to find a way to find z as a function of x and y, but we have a normal vector given to us, then we have another formula which says that ndS is, sorry, I should have said it's always up to sign because we have a two orientation convention. We have to decide based on what we are trying to do, whether we are doing the correct convention or the wrong one. So, the other formula is n over n dot k dx dy. Sorry, are they all the same? Well, if you want, you can put an absolute value here. But, it doesn't matter because it's up to sign anyway. So, I mean, this formula is valid as it is. OK, and, I mean, if you're in a situation where you can apply more than one formula, they will all give you the same answer in the end because it's the same flux integral. OK, so anyway, so we have various ways of computing surface integrals, and probably one of the best possible things you can do to prepare for the test is actually to look again at some practice problems from the notes that do flux integrals over various kinds of surfaces because that's probably one of the hardest topics in this unit of the class. OK, anyway, let's move on to line integrals. So, those are actually a piece of cake in comparison, OK, because all that this is, is just integral of P dx Q dy R dz. And, then all you have to do is parameterize the curve, C, to express everything in terms of a single variable. And then, you end up with a usual single integral, and you can just compute it. So, that one works pretty much as it did in the plane. So, if you forgotten what we did in the plane, it's really the same thing. OK, so now we have three different kinds of integrals, and really, well, they certainly have in common that they integrate things somehow. But, apart from that, they are extremely different in what they do. I mean, this one involves a function, a scalar quantity. These involve vector quantities. They don't involve the same kinds of shapes over which to integrate. Here, you integrate over a three-dimensional region. Here, you integrate only over a two-dimensional surface, and here, only a one-dimensional curve. So, try not to confuse them. That's basically the most important advice. Don't get mistaken. Each of them has a different way of getting evaluated. Eventually, they will all give you numbers, but through different processes. So now, well, I said these guys are completely different. Well, they are, but we still have some bridges between them. OK, so we have two, maybe I should say three, well, two bridges between these guys. OK, so we have somehow a connection between these which is the divergence theorem. We have a connection between that, which is Stokes theorem. So -- Just to write them again, so the divergence theorem says if I have a region in space, and I call its boundary S, so, it's going to be a closed surface, and I orient S with a normal vector pointing outwards, then whenever I have a surface integral over S, sorry, I can replace it by a triple integral over the region inside. OK, so this guy is a vector field. And, this guy is a function that somehow relates to the vector field. I mean, you should know how. You should know the definition of divergence, of course. But, what I want to point out is if you have to compute the two sides separately, well, this is just, you know, your standard flux integral. This is just your standard triple integral over a region in space. Once you have computed what this guy is, it's really just a triple integral of the function. So, the way in which you compute it doesn't see that it came from a divergence. It's just the same way that you would compute any other triple integral. The way we compute it doesn't depend on what actually we are integrating. Stokes theorem says if I have a curve that's the boundary of a surface, S, and I orient the two in compatible manners, then I can replace a line integral on C by a surface integral on S. OK, and that surface integral, well, it's not for the same vector field. This relates a line integral for one field to a surface integral from another field. That other field is given from the first one just by taking its curl So, after you take the curl, you obtain a different vector field. And, the way in which you would compute the surface integral is just as with any surface integral. You just find a formula for ndS dot product, substitute, evaluate. The calculation of this thing, once you've computed curl does not remember that it was a curl.

It's the same as with any other flux integral. OK, and finally, the last bridge, so this was between two and three. This was between one and two. Let me just say, there's a bridge between zero and one, which is that if you have a function in its gradient, well, the fundamental theorem of calculus says that the line integral for the vector field given by the gradient of a function is actually equal to the change in value of a function. That's if you have a curve bounded by P0 and P1. So in a way, actually, each of these three theorems relates a quantity with a certain number of integral signs to a quantity with one more integral sign. And, that's actually somehow a fundamental similarity between them. But maybe it's easier to think of them as completely different stories. So now, with this one, we additionally have to remember another topic is given a vector field, F, with curl equal to zero, find the potential. And, we've seen two methods for that, and I'm sure you remember them. So, if not, then try to remember them for Tuesday. OK, so anyway, again, conceptually, we have, really, three different kinds of integrals. We evaluated them in completely different ways, and we have a handful of theorems, connecting them to each other. But, that doesn't have any impact on how we actually compute things.