18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 18. – **Tue, Oct 23, 2007**

Change of variables.

Example 1: area of ellipse with semiaxes a and b: setting $u = x/a$, $v = y/b$,

$$\iint_{(x/a)^2+(y/b)^2<1} dx\,dy = \iint_{u^2+v^2<1} ab\,du\,dv = ab\iint_{u^2+v^2<1} du\,dv = \pi ab.$$

(substitution works here as in 1-variable calculus: $du = \frac{1}{a}\,dx$, $dv = \frac{1}{b}\,dy$, so $du\,dv = \frac{1}{ab}dx\,dy$.

In general, must find out the scale factor (ratio between $du\,dv$ and $dx\,dy$)?

Example 2: say we set $u = 3x-2y$, $v = x+y$ to simplify either integrand or bounds of integration. What is the relation between $dA = dx\,dy$ and $dA' = du\,dv$? (area elements in xy- and uv-planes).

Answer: consider a small rectangle of area $\Delta A = \Delta x\Delta y$, it becomes in uv-coordinates a parallelogram of area $\Delta A'$. Here the answer is independent of which rectangle we take, so we can take e.g. the unit square in xy-coordinates.

In the uv-plane, $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, so this becomes a parallelogram with sides given by vectors $\langle 3,1\rangle$ and $\langle -2,1\rangle$ (picture drawn), and area = det = $\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} = 5 \left(= \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \right)$.

For any rectangle $\Delta A' = 5\Delta A$, in the limit $dA' = 5dA$, i.e. $du\,dv = 5dx\,dy$. So $\iint \dots dx\,dy = \iint \dots \frac{1}{5}du\,dv$.

General case: approximation formula $\Delta u \approx u_x\Delta x + u_y\Delta y$, $\Delta v \approx v_x\Delta x + v_y\Delta y$, i.e.

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

A small xy-rectangle is approx. a parallelogram in uv-coords, but scale factor depends on x and y now. By the same argument as before, the scale factor is the determinant.

Definition: the Jacobian is $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$. Then $du\,dv = |J|\,dx\,dy$.

(absolute value because area is the absolute value of the determinant).

Example 1: polar coordinates $x = r\cos\theta$, $y = r\sin\theta$:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

So $dx\,dy = r\,dr\,d\theta$, as seen before.

Example 2: compute $\int_0^1\int_0^1 x^2y\,dx\,dy$ by changing to $u = x$, $v = xy$ (usually motivation is to simplify either integrand or region; here neither happens, but we just illustrate the general method).

1) Area element: Jacobian is $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x$, so $du\,dv = x\,dx\,dy$, i.e. $dx\,dy = \frac{1}{x}du\,dv$.

2) Express integrand in terms of u, v: $x^2y\,dx\,dy = x^2y\,\frac{1}{x}\,du\,dv = xy\,du\,dv = v\,du\,dv$.

3) Find bounds (picture drawn): if we integrate $du\,dv$, then first we keep $v = xy$ constant, slice looks like portion of hyperbola (picture shown), parametrized by $u = x$. The bounds are: at the top boundary $y = 1$, so $v/u = 1$, i.e. $u = v$; at the right boundary, $x = 1$, so $u = 1$. So the inner

integral is $\int_v^1$. The first slice is $v = 0$, the last is $v = 1$; so we get

$$\int_0^1 \int_v^1 v\,du\,dv.$$

Besides the picture in xy coordinates (a square sliced by hyperbolas), I also drew a picture in uv coordinates (a triangle), which some students may find is an easier way of getting the bounds for u and v.

18.02 Lecture 19. – **Thu, Oct 25, 2007**

Handouts: PS7 solutions; PS8.

Vector fields.

$\vec{F} = M\hat{\imath} + N\hat{\jmath}$, where $M = M(x, y)$, $N = N(x, y)$: at each point in the plane we have a vector $\vec{F}$ which depends on x, y.

Examples: velocity fields, e.g. wind flow (shown: chart of winds over Pacific ocean); force fields, e.g. gravitational field.

Examples drawn on blackboard: (1) $\vec{F} = 2\hat{\imath} + \hat{\jmath}$ (constant vector field); (2) $\vec{F} = x\hat{\imath}$; (3) $\vec{F} = x\hat{\imath} + y\hat{\jmath}$ (radially outwards); (4) $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ (explained using that $\langle -y, x\rangle$ is $\langle x, y\rangle$ rotated 90° counterclockwise).

Work and line integrals.

$W = (\text{force}).(\text{distance}) = \vec{F} \cdot \Delta\vec{r}$ for a small motion $\Delta\vec{r}$. Total work is obtained by summing these along a trajectory C: get a "line integral"

$$W = \int_C \vec{F} \cdot d\vec{r} \ \left(= \lim_{\Delta\vec{r} \to 0} \sum_i \vec{F} \cdot \Delta\vec{r}_i \right).$$

To evaluate the line integral, we observe C is parametrized by time, and give meaning to the notation $\int_C \vec{F} \cdot d\vec{r}$ by

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} (\vec{F} \cdot \frac{d\vec{r}}{dt})\, dt.$$

Example: $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$, C is given by $x = t$, $y = t^2$, $0 \le t \le 1$ (portion of parabola $y = x^2$ from (0,0) to (1,1)). Then we substitute expressions in terms of t everywhere:

$$\vec{F} = \langle -y, x\rangle = \langle -t^2, t\rangle, \quad \frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}\rangle = \langle 1, 2t\rangle,$$

so $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt = \int_0^1 \langle -t^2, t\rangle \cdot \langle 1, 2t\rangle\, dt = \int_0^1 t^2\, dt = \frac{1}{3}$. (in the end things always reduce to a one-variable integral.)

In fact, the definition of the line integral does not involve the parametrization: so the result is the same no matter which parametrization we choose. For example we could choose to parametrize the parabola by $x = \sin\theta$, $y = \sin^2\theta$, $0 \le \theta \le \pi/2$. Then we'd get $\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \dots\, d\theta$, which would be equivalent to the previous one under the substitution $t = \sin\theta$ and would again be equal to $\frac{1}{3}$. In practice we always choose the simplest parametrization!

New notation for line integral: $\vec{F} = \langle M, N\rangle$, and $d\vec{r} = \langle dx, dy\rangle$ (this is in fact a differential: if we divide both sides by dt we get the component formula for the velocity $d\vec{r}/dt$). So the line integral becomes

$$\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy.$$

The notation is dangerous: this is not a sum of integrals w.r.t. x and y, but really a line integral along C. To evaluate one must express everything in terms of the chosen parameter.

In the above example, we have $x = t$, $y = t^2$, so $dx = dt$, $dy = 2t\,dt$ by implicit differentiation; then

$$\int_C -y\,dx + x\,dy = \int_0^1 -t^2\,dt + t\,(2t)\,dt = \int_0^1 t^2\,dt = \frac{1}{3}$$

(same calculation as before, using different notation).

Geometric approach.

Recall velocity is $\dfrac{d\vec{r}}{dt} = \dfrac{ds}{dt}\,\hat{\boldsymbol{T}}$ (where $s=$arclength, $\hat{\boldsymbol{T}}=$unit tangent vector to trajectory).

So $d\vec{r} = \hat{\boldsymbol{T}}\,ds$, and $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{\boldsymbol{T}}\,ds$. Sometimes the calculation is easier this way!

Example: $C=$circle of radius a centered at origin, $\vec{F} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$, then $\vec{F}\cdot\hat{\boldsymbol{T}} = 0$ (picture drawn), so $\int_C \vec{F}\cdot\hat{\boldsymbol{T}}\,ds = \int 0\,ds = 0$.

Example: same C, $\vec{F} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$, then $\vec{F}\cdot\hat{\boldsymbol{T}} = |\vec{F}| = a$, so $\int_C \vec{F}\cdot\hat{\boldsymbol{T}}\,ds = \int a\,ds = a\,(2\pi a) = 2\pi a^2$; checked that we get the same answer if we compute using parametrization $x = a\cos\theta$, $y = a\sin\theta$.

18.02 Lecture 20. – **Fri, Oct 26, 2007**

Line integrals continued.

Recall: line integral of $\vec{F} = M\hat{\boldsymbol{\imath}} + N\hat{\boldsymbol{\jmath}}$ along a curve C: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy = \int_C \vec{F}\cdot\hat{\boldsymbol{T}}\,ds$.

Example: $\vec{F} = y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$, $\int_C \vec{F}\cdot d\vec{r}$ for $C = C_1 + C_2 + C_3$ enclosing sector of unit disk from 0 to $\pi/4$. (picture shown). Need to compute $\int_{C_i} y\,dx + x\,dy$ for each portion:

1) x-axis: $x = t$, $y = 0$, $dx = dt$, $dy = 0$, $0 \le t \le 1$, so $\int_{C_1} y\,dx + x\,dy = \int_0^1 0\,dt = 0$. Equivalently, geometrically: along x-axis, $y = 0$ so $\vec{F} = x\hat{\boldsymbol{\jmath}}$ while $\hat{\boldsymbol{T}} = \hat{\boldsymbol{\imath}}$ so $\int_{C_1} \vec{F}\cdot\hat{\boldsymbol{T}}\,ds = 0$.

2) C_2: $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$, $0 \le \theta \le \frac{\pi}{4}$. So

$$\int_{C_2} y\,dx + x\,dy = \int_0^{\pi/4} \sin\theta(-\sin\theta)d\theta + \cos\theta\cos\theta\,d\theta = \int_0^{\pi/4} \cos(2\theta)d\theta = \left[\frac{1}{2}\sin(2\theta)\right]_0^{\pi/4} = \frac{1}{2}.$$

3) C_3: line segment from $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ to $(0,0)$: could take $x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t$, $y=$same, $0 \le t \le 1$, ... but easier: C_3 backwards ("$-C_3$") is $y = x = t$, $0 \le t \le \frac{1}{\sqrt{2}}$. Work along $-C_3$ is opposite of work along C_3.

$$\int_{C_3} y\,dx + x\,dy = \int_{1/\sqrt{2}}^0 t\,dt + t\,dt = -\int_0^{1/\sqrt{2}} 2t\,dt = -[t^2]_0^{1/\sqrt{2}} = -\frac{1}{2}.$$

If $\vec{F}$ is a gradient field, $\vec{F} = \nabla f = f_x\hat{\boldsymbol{\imath}} + f_y\hat{\boldsymbol{\jmath}}$ (f is called "potential function"), then we can simplify evaluation of line integrals by using the fundamental theorem of calculus.

Fundamental theorem of calculus for line integrals:

$\displaystyle\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$ when C runs from P_0 to P_1.

Equivalently with differentials: $\displaystyle\int_C f_x\,dx + f_y\,dy = \int_C df = f(P_1) - f(P_0)$. Proof:

$$\int_C \nabla f \cdot d\vec{r} = \int_{t_0}^{t_1} (f_x \frac{dx}{dt} + f_y \frac{dy}{dt})\,dt = \int_{t_0}^{t_1} \frac{d}{dt}(f(x(t), y(t))\,dt = [f(x(t), y(t))]_{t_0}^{t_1} = f(P_1) - f(P_0).$$

E.g., in the above example, if we set $f(x,y) = xy$ then $\nabla f = \langle y, x\rangle = \vec{F}$. So $\int_{C_i}$ can be calculated just by evaluating $f = xy$ at end points. Picture shown of C, vector field, and level curves.

Consequences: for a gradient field, we have:

• Path independence: if C_1, C_2 have same endpoints then $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$ (both equal to $f(P_1) - f(P_0)$ by the theorem). So the line integral $\int_C \nabla f \cdot d\vec{r}$ depends only on the end points, not on the actual trajectory.

• Conservativeness: if C is a closed loop then $\int_C \nabla f \cdot d\vec{r} = 0$ $(= f(P) - f(P))$.
(e.g. in above example, $\int_C = 0 + \frac{1}{2} - \frac{1}{2} = 0$.)

WARNING: this is only for gradient fields!

Example: $\vec{F} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$ is not a gradient field: as seen Thursday, along C = circle of radius a counterclockwise ($\vec{F}//\hat{\boldsymbol{T}}$), $\int_C \vec{F} \cdot d\vec{r} = 2\pi a^2$. Hence $\vec{F}$ is not conservative, and not a gradient field.

Physical interpretation.

If the force field $\vec{F}$ is the gradient of a potential f, then work of $\vec{F}$ = change in value of potential.

E.g.: 1) $\vec{F}$ = gravitational field, f = gravitational potential; 2) $\vec{F}$ = electrical field; f = electrical potential (voltage). (Actually physicists use the opposite sign convention, $\vec{F} = -\nabla f$).

Conservativeness means that energy comes from change in potential f, so no energy can be extracted from motion along a closed trajectory (conservativeness = conservation of energy: the change in kinetic energy equals the work of the force equals the change in potential energy).

We have four equivalent properties:

(1) $\vec{F}$ is conservative ($\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C)

(2) $\int F \cdot d\vec{r}$ is path independent (same work if same end points)

(3) $\vec{F}$ is a gradient field: $\vec{F} = \nabla f = f_x\hat{\boldsymbol{\imath}} + f_y\hat{\boldsymbol{\jmath}}$.

(4) $M\,dx + N\,dy$ is an exact differential $(= f_x\,dx + f_y\,dy = df.)$

((1) is equivalent to (2) by considering C_1, C_2 with same endpoints, $C = C_1 - C_2$ is a closed loop. (3) $\Rightarrow$ (2) is the FTC, $\Leftarrow$ will be key to finding potential function: if we have path independence then we can get $f(x,y)$ by computing $\int_{(0,0)}^{(x,y)} \vec{F} \cdot d\vec{r}$. (3) and (4) are reformulations of the same property).