Hyperbolic Sine

In this problem we study the hyperbolic sine function:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

reviewing techniques from several parts of the course.

a) Sketch the graph of $y = \sinh x$ by finding its critical points, points of inflection, symmetries, and limits as $x \to \infty$ and $-\infty$.

b) Give a suitable definition for $\sinh^{-1} x$ (the inverse hyperbolic sine) and sketch its graph, indicating the domain of definition.

c) Find $\dfrac{d}{dx} \sinh^{-1} x$.

d) Use your work to evaluate $\displaystyle\int \frac{dx}{\sqrt{a^2 + x^2}}$.

Solution

This problem provides a nice review of a variety of material covered in the course and also explains why electronic integration tools frequently give answers that involve the hyperbolic sine function.

a) Sketch the graph of $y = \sinh x$ by finding its critical points, points of inflection, symmetries, and limits as $x \to \infty$ and $-\infty$.

   We need to test for symmetry, take derivatives, and take limits. We can perform these tasks in any order; here they're presented in the order they are listed in the problem.

   To find the critical points of $\sinh x$ we take its first derivative and set it equal to zero.

$$\begin{aligned} \frac{d}{dx} \sinh x &= \frac{d}{dx} \frac{1}{2}(e^x - e^{-x}) \\ &= \frac{1}{2}(e^x + e^{-x}) \\ &= \cosh x \end{aligned}$$

   Because e^x is always positive, the derivative of $\sinh x$ is never 0 and so $\sinh x$ has no critical points.

   To find points of inflection we set the second derivative equal to 0.

$$\begin{aligned} \frac{d^2}{dx^2} \sinh x &= \frac{d}{dx} \frac{1}{2}(e^x + e^{-x}) \\ &= \frac{1}{2}(e^x - e^{-x}) \\ &= \sinh x \end{aligned}$$

The second derivative of $\sinh x$ equals 0 when $e^x = e^{-x}$, or when $x = 0$. To confirm that this is an inflection point, we test the value of the second derivative of $\sinh x$ to the left and right of this point:

$$\begin{aligned} y''(1) = \sinh(1) &= \frac{1}{2}(e^1 - e^{-1}) = \frac{1}{2}(e - \frac{1}{e}) > 1 \\ y'(-1) = \sinh(-1) &= \frac{1}{2}(e^{-1} - e^1) = \frac{1}{2}(\frac{1}{e} - e) < 1 \end{aligned}$$

We conclude that the graph of $y = \sinh x$ has no critical points and an inflection point at $x = 0$.

To find the symmetries of a function, we find the output of the function when $-x$ is input.

$$\begin{aligned} \sinh(-x) &= \frac{1}{2}(e^{-x} - e^x) \\ &= -\frac{1}{2}(e^x - e^{-x}) \\ &= -\sinh x \end{aligned}$$

If $f(-x) = -f(x)$ we say that $f(x)$ is an odd function; the graph of $y = \sinh x$ is symmetric about the origin.

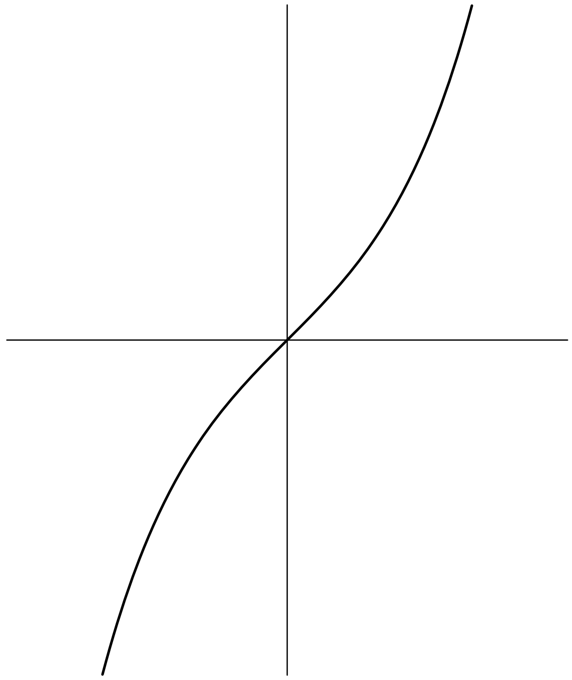
Recall that $\sinh x = \frac{1}{2}(e^x - e^{-x})$. As x approaches positive infinity, the value of e^x approaches positive infinity and the value of e^{-x} approaches zero.

$$\lim_{x \to \infty} \sinh x = \infty$$

Similarly, as x approaches $-\infty$ the value of e^{-x} dominates and e^x vanishes.

$$\lim_{x \to -\infty} \sinh x = -\infty$$

We now have a good deal of information about the graph, including the values of $\sinh(1)$ and $\sinh(-1)$. In addition, we can easily determine that $\sinh(0) = 0$. Our graph will look something like:



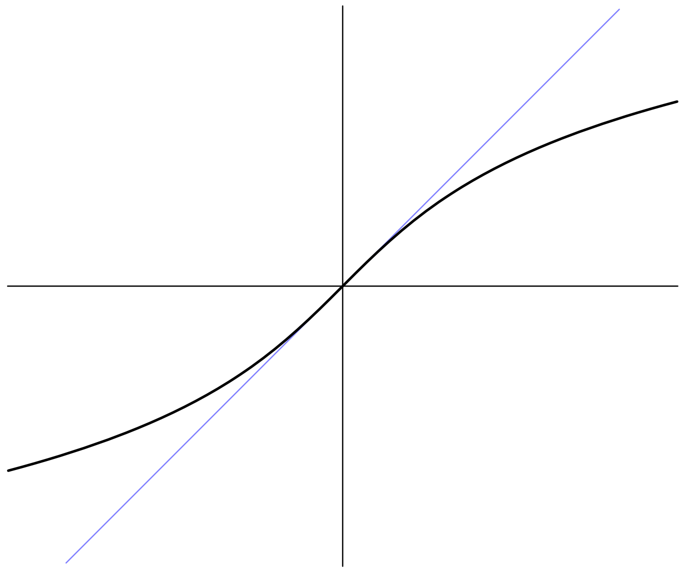
b) Give a suitable definition for $\sinh^{-1} x$ (the inverse hyperbolic sine) and sketch its graph, indicating the domain of definition.

While it's possible to find an algebraic expression for the inverse of $\sinh x$, it's not necessary for this problem.

We can tell from the graph of $\sinh x$ that the function is invertible, so we can apply the definition of the inverse of a function:

$$y = \sinh^{-1} x \text{ is the unique value of } y \text{ for which } x = \sinh y.$$

The graph of $y = \sinh^1 x$ is the reflection of the graph of $y = \sinh x$ across the line $y = x$.



The domain of $\sinh^{-1} x$ is all real numbers.

c) Find $\dfrac{d}{dx} \sinh^{-1} x$.

To answer this we must review material from the first unit of the semester! We learned how to use implicit differentiation to find the derivative $\frac{dy}{dx}$ of the inverse of a function.

$$\begin{aligned} y &= \sinh^{-1} x \\ \sinh y &= x \\ \frac{d}{dx} \sinh y &= \frac{d}{dx} x \\ (\cosh y) \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cosh y} \end{aligned}$$

At this point we wish to rewrite $\cosh y$ in terms of x. When dealing with trigonometric functions we could use a right triangle and the Pythagorean theorem for this task; here we must rely on hyperbolic trig identities. (If you need these identities on the final exam, they will be given to you.) Using the following identity, we can transform an expression in terms of $\cosh y$ to one in terms of $\sinh y = x$.

$$\cosh^2 t - \sinh^2 t = 1$$

$$\begin{aligned}\cosh^2 t &= 1+\sinh^2 t\\ \cosh t &= \sqrt{1+\sinh^2 t}\end{aligned}$$

Thus:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\cosh y}\\ &= \frac{1}{\sqrt{1+\sinh^2 y}}\\ \frac{dy}{dx} &= \frac{1}{\sqrt{1+x^2}}.\end{aligned}$$

We can get an idea of whether our work is correct by noting that this equation should describe the slope of the tangent line to the graph of $\sinh^{-1} x$.

d) Use your work to evaluate $\int \frac{dx}{\sqrt{a^2+x^2}}$.

We can reinterpret our results from the previous problem to say that:

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + c.$$

This gives us an alternative to the trig substitution $x = a\tan\theta$ when dealing with integrands of the form $\frac{dx}{\sqrt{a^2+x^2}}$.

$$\begin{aligned}\int \frac{dx}{\sqrt{a^2+x^2}} &= \int \frac{a\,du}{\sqrt{a^2+(au)^2}} \qquad \text{(Substitute } x = au\text{)}\\ &= \int \frac{du}{\sqrt{1+u^2}}\\ &= \sinh^{-1} u + c\\ &= \sinh^{-1}\left(\frac{x}{a}\right) + c\end{aligned}$$

Note that this is much simpler than the calculation involving the substitution $x = a\tan\theta$.