

JOEL LEWIS: Hi. Welcome back to recitation. We've been talking about Taylor series and different sorts of manipulations you can do with them, and different examples of Taylor series. So I have an example here that I don't think you've seen in lecture. So this is the Taylor series $1 + 2x + 3x^2 + 4x^3 + 5x^4$, and so on.

I'm going to tell you that this is a Taylor series for a fairly nice function. And what I'd like you to do, is try and figure out what that function is. Now, I'm kind of sending you off onto this task without giving you much guidance, so let me give you a little bit of a hint, which is that the thing to do here, is you have a bunch of different tools that you know how to manipulate Taylor series by. So you know how to, you know, do calculus on Taylor series, you can take derivatives and integrals, you know how to add and multiply and perform these sorts of manipulations. So you might think of a manipulation you could perform on this Taylor series that'll make it a simpler expression, or something you're already familiar with, or so on.

So let me give you some time to work on that. Think about it for a while. Try and come up with the expression for this Taylor series. And come back, and we can work on it together.

So hopefully you had some luck working on this problem, and figured out what the right manipulation to use is. I didn't-- I kind of tossed it at you without a whole lot of guidance, and I don't think you've done a lot of examples like this before. So it's a little tricky.

So what I suggested was that you think about things that you know how to do that could be used to simplify this. So looking at it, it's just one Taylor series. So it's not clear that sort of Taylor series arithmetic is going to help you very much. It's not obviously a substitution of some value into some other Taylor series that you're familiar with. It's not obviously, say, a sum of two Taylor series that you already know very well.

So those things don't, aren't immediately, it's not clear where to go with them. One thing that does pop out-- well, OK. So let's talk about some of the other tools that I mentioned. We have calculus on Taylor series. So we have differentiation. And if you take-- so we see here that the coefficients are sort of a linear polynomial. If you take a derivative, what happens is, well, this becomes $2 + 6x + 12x^2 + 20x^3$. And those coefficients, 2, 6, 12, 20, those are given by a quadratic polynomial. So that makes our life kind of more complicated, somehow.

But if we look at this, we see that integrating this power series is really easy to do. This power series has a really nice antiderivative. So what I'm going to do, is I'm going to call this power series by the name f of x . And then what I'd like you to notice, is that the antiderivative of f of x dx -- well, we can integrate power series termwise. And so what we get is-- well, so all right. So I'm going to do-- I'm going to put the constant of integration first. So the antiderivative of this is c plus-- well, 1 , you take its integral and you get x . And $2x$, you take its integral, and you just get plus x squared. And $3x$ squared, you take its integral and you get plus x cubed, and plus x to the fourth from the next one, and plus x to the fifth, and so on.

So, OK. So I put the c here, right? So any power series of this form is an antiderivative of the power series that we started with. Any power series of this form has the derivative equal to thing that we're interested in. And since we really care about what f is, and not what its antiderivative is, we can choose c to be any convenient value.

So I'm gonna, in particular, look at the power series where c is equal to 1 . And why am I going to make that choice? Well, because we've seen a power series that looks very much like this before, with the 1 there. So we know $1 + x + x^2 + x^3 + x^4 + \dots$ and so on. We know that this is equal to $\frac{1}{1-x}$. OK? So since we know that this is the case, that means that our power series, f of x , is just the derivative of this. That's what we just showed here. So, f of x is equal to $\frac{d}{dx}$ of $\frac{1}{1-x}$ whoops that should be $\frac{1}{1-x}$. OK? And notice that, you know, if I'd chosen a different choice of constant here, it would be killed off by this differentiation. So it really was irrelevant.

So $\frac{d}{dx}$ of $\frac{1}{1-x}$. Yes, of $\frac{d}{dx}$ of $\frac{1}{1-x}$. And, OK. Well, this is an easy derivative to compute. This is $\frac{1}{1-x}$ to the minus 1 , so you do a little chain rule thing, and I think what you get out is that this is $\frac{1}{(1-x)^2}$.

So this gives me a nice formula for this function f of x . If you wanted to check, one thing you could do, is you could set about computing a few terms, the power series for this function, for $\frac{1}{(1-x)^2}$, $\frac{1}{(1-x)^2}$ I should say. So you could do that either by using your derivative formula, which is easy enough to do. Another thing you could do, is you could realize that this is $\frac{1}{1-x}$ times $\frac{1}{1-x}$, so you could try multiplying that polynomial by itself and that power series by itself, and see if it's easy to get back this formula that we had.

But any of those ways is a good way to double check that this is really true, that this function,

that this power series here, f of x , has this functional form.

Now of course, I haven't said anything about the radius of convergence, but the thing to remember when you're doing calculus on power series, is that when you take a derivative or an antiderivative of a power series, you don't change the radius of convergence. Sometimes you can fiddle with what happens at the endpoints, but the radius of convergence stays the same. So this is going to be true whenever x is between negative 1 and 1. And in fact, this series diverges at this endpoint. That's pretty easy to check.

So there we go. So this is the functional form for this power series. It's valid whenever x is between negative 1 and 1. That's its range of convergence. And so here we have a cute little trick for figuring out some forms of power series.

And of course if you were interested, so like I said, now you had a-- you could look at the derivative of this, which I mentioned had some coefficients that were given by some quadratic polynomial. Now that you have a functional form, you could figure out, you know, "Oh what, you know, what is the function whose power series has that quadratic polynomial as coefficients?" And you can do a whole bunch of other stuff by similarly taking derivatives of other power series that you're familiar with, or integrals.

So I'll stop there.