Operations on Power Series Related to Taylor Series

In this problem, we perform elementary operations on Taylor series – term by term differentiation and integration – to obtain new examples of power series for which we know their sum. Suppose that a function f has a power series representation of the form:

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x-c)^n$$

convergent on the interval $(c-R, c+R)$ for some R. The results we use in this example are:

- (Differentiation) Given f as above, $f'(x)$ has a power series expansion obtained by by differentiating each term in the expansion of $f(x)$:

$$f'(x) = a_1 + a_2(x-c) + 2a_3(x-c) + \cdots = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$

- (Integration) Given f as above, $\int f(x)\,dx$ has a power series expansion obtained by by integrating each term in the expansion of $f(x)$:

$$\int f(x)\,dx = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \cdots = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

  for some constant C depending on the choice of antiderivative of f.

Questions:

1. Find a power series representation for the function $f(x) = \arctan(5x)$. (Note: $\arctan x$ is the inverse function to $\tan x$.)
2. Use power series to approximate
$$\int_0^1 \sin(x^2)\,dx$$
   (Note: $\sin(x^2)$ is a function whose antiderivative is not an elementary function.)