

PROFESSOR: Hi. Welcome back to recitation. You've been talking about computing limits of some indeterminate forms. In particular, you used l'Hopital's rule to help you out with some limits in the form 0 over 0 or infinity over infinity. So these are the two indeterminate ratios. And you also-- so, OK so when you have a limit that is in the form of an indeterminate ratio, you've seen that one tool that you can use to help compute the limit is l'Hopital's rule. There are other indeterminate forms for limits as well. So you actually saw, in lecture, another one of these, which was you saw a limit of the form 0 to the 0. So that was the limit you saw, was the limit as x goes to 0 from the right of x to the x . So that was going to 0 over 0.

And the two competing forces are that as the base of the limit goes to 0, that wants to make the whole thing get closer to 0. And when the exponent is going to 0, that makes the whole thing want to get closer to 1. So you have those two competing forces. That's why it is an indeterminate form.

When you were solving this limit, you-- first thing you did was you wrote it as an exponential in base e . So you wrote x to the x as e to the $x \ln x$. So $x \ln x$ is also an indeterminate form as x goes to 0. It's an indeterminate form of the form 0 times infinity. Here I'm writing infinity to mean either positive infinity or negative infinity.

So in this case, this is an indeterminate form because when you have one factor going to 0, that makes the whole product want to get closer to 0. Whereas when you have one factor going to infinity, either positive or negative, that makes the whole product want to get big. So that's why it becomes indeterminate. And you were able to evaluate the limit $x \ln x$, in that case, by rewriting it as a quotient.

There are a few other indeterminate forms that I'd like to mention. So in particular, there are three other indeterminate forms different from these. And then I'll give you an example of one of them to solve a problem. So the three other indeterminate forms, there are two more that are sort of an exponential indeterminate form, and there's one sort of outlier.

So one of the other indeterminate forms is infinity to the 0. So when you have an exponential expression where the base is getting very, very large, and the exponent is going to 0, well the base getting large makes the whole fraction want to be-- sorry-- the whole expression want to be large, whereas the exponent going to 0 makes the whole expression want to get closer to 1. And so those two forces are in tension, and you can end up with limits that equal any value

when you have a limit like this, infinity to the 0.

Another similar one is 1 to the infinity. And this is the one I'll give you an exercise on in a minute. So when you have a limit of the form 1 to the infinity, so something to the something where the base is going to 1 and the exponent is going to infinity, the base going to 1 makes this whole thing want to go to 1. The exponent to infinity makes this whole thing want to either blow up if it's a little bigger than 1, or get really small if it's a little smaller than 1. So you have, again, a tension there, and the result is indeterminate.

The one sort of unusual one that you have is also-- which I'm not going to talk much more about-- is infinity minus infinity. So when you have two very large things, so here I mean either positive infinity minus positive infinity, or negative infinity minus negative infinity. When you have two things that are both getting very large, their difference could also be getting very large, or it could be getting very small, or could be doing anything in between. So that's also an indeterminate form.

So these are the seven indeterminate forms. When you have a quotient, you can always apply l'Hopital's rule. When you have a product, you can always rewrite it as a quotient, by writing for example, $x^2 + 1$ times-- well, let's see-- e^{-x} . That's a product. And you could always rewrite it as a quotient, for example, e^{-x} over $1/(x^2 + 1)$. There might even be a smarter way to rewrite this product as a quotient.

But, OK. And so when you have an exponential, as in this case, you can use this rewriting in base e trick to turn it into a product, which you can then turn into a quotient. For the difference case, the reason I say it's unusual, is just that there's not a good, general method for working with these. There are a lot of special cases. And you sort of have to-- or what I mean is you have to analyze them on a case by case basis. There are different sort of techniques that will work.

So let me give you an example of one of these 1 to the infinity kinds. So why don't we come over here. So compute the limit as x goes to 0 from the right of $1 + 3x$ to the 10 divided by x . So we see as x is going to 0, this base is going to 1. And so that makes this whole expression want to be close to 1, whereas this exponent is going to infinity, since this is just a little bit bigger than 1, that makes the whole expression want to be big when the exponent is big. So you have a tension here between the base going to 1 and the exponent going to infinity.

So the question is, try and actually compute this limit. So why don't you pause the video, take a couple of minutes to work on this question, come back, and we can work on it together.

Welcome back. Hopefully you had some luck working on this problem. As we said, this is a limit of an indeterminate form of the 1 to infinity type. As one of the three exponential types of indeterminate forms, a really promising first step almost every time is to rewrite this as an exponential expression with the base e . So we just do-- first we just do an algebraic manipulation on the thing we're taking the limit of, and then often, very often, that simplifies it into something that we can actually compute the limit of.

So in particular, we have that $1 + 3x$, if we want to write this in exponential form, this is equal to e to the \ln of $1 + 3x$. This is true of any positive number. Any positive number is e to the \ln of it, because e and \log are inverse functions.

So OK, so we write it like this. And so that means that $(1 + 3x)^{10/x}$ is equal to e to the \ln of $(1 + 3x)^{10/x}$. And now you can use your exponent rules. So this is equal to e to the \ln of $1 + 3x$ times $10/x$. So our original expression is equal to this down here. So the limit of our original expression is equal to the limit of this one. The other thing to notice is that because exponentiation is a nice, continuous function, in order to compute this limit, or the limit of this expression, it suffices-- just with base a constant, e -- it suffices to compute the limit of the exponent.

All right, so let's do that. Let's compute the limit of the exponent. So we have the limit, so it has to be x going to the same place, which in this case is to 0 from the right, of \ln of $1 + 3x$ times $10/x$. Well, this is a product. It came to us as a product. But there's an obvious way to rewrite this as a quotient. I should say it's an indeterminate product. As x goes to 0, this goes to \ln of 1, which is 0. Whereas this goes to infinity, positive infinity since we're coming from the right. So this is a 0 times positive infinity form. So it is an indeterminate limit-- or sorry-- an indeterminate product. And, right, and there's an obvious way to rewrite this as an indeterminate quotient, which is to rewrite it as the limit as x goes 0 plus of $10 \ln$ of $1 + 3x$ divided by x , as x goes 0.

So now, this is a limit where it's an infinity-- sorry-- a 0 over 0 form. So OK, so good. So now we can apply l'Hopital's rule. So by l'Hopital's rule, this is equal to the limit as x goes to 0 on the right of-- well, we apply l'Hopital's rule on the top, we get 10 over $1 + 3x$, by the chain rule, times 3. And on the bottom, we just get one. So now, OK, so this is true provided this

second limit exists. And the second limit is no longer indeterminate. It's easy to see what it is. You just plug in x equals 0, and we get that this is equal to 30.

OK so this limit is equal to 30, but this isn't the limit that we started out wanting to compute. The limit we started out wanting to compute is e to the \ln of $1 + 3x$ times 10 over x . So it's e to the this. So our original limit as x goes to 0 from the right of $1 + 3x$ to the 10 over x is equal to e to the thirtieth power, which is pretty huge. OK, so that's how we-- right-- and OK, so that's the answer to our question.

So we took our original limit, it was this indeterminate exponential form. So what we do to it is, when you have an indeterminate exponential form, you do this rewriting as an exponential in the base of e trick, and then you pass the limit into the exponent.

Because e is now a nice constant, life is simple-- life is simpler I should say-- you pass the limit into the constant, then you have an indeterminate ratio, indeterminate product, which you rewrite as an indeterminate ratio on which you can then apply l'Hopital's rule. Or possibly, you know, you rewrite it as a product. Then it's easy to see what the value is.

So, all right, so that's how we deal with limits of indeterminate exponential forms. And I'll end there.