Rate of Growth of $\ln x$

This expression is in indeterminate form but looks like it might be the wrong type. This isn't a fraction, so we have to think about how to apply l'Hôpital's rule.

In the expression, the factor x is approaching 0 while the factor $\ln x$ is approaching negative infinity.

$$\lim_{x\to 0^+} \underbrace{x}_{\to 0} \underbrace{\ln x}_{\to -\infty}$$

We're multiplying a number that's getting smaller and smaller by one that's getting larger and larger; the result could be really large or really small, depending on rates of growth.

The first step in finding the limit is to rewrite the expression as a ratio, rather than as a product. We'll choose to write it as:

$$x \ln x = \frac{\ln x}{1/x}.$$

This is an expression of the type $\frac{-\infty}{\infty}$, which is one of the forms we can apply l'Hôpital's rule to. Let's do that:

$$\begin{aligned} \lim_{x\to 0^+} x \ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x} \\ &= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \qquad \text{(l'Hop)} \\ &= \lim_{x\to 0^+} -x \\ &= 0. \end{aligned}$$

We conclude that x goes to 0 faster than $\ln x$ goes to negative infinity, and so the limit of the product is 0. You might not have been able to guess this in advance.

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