Polar Coordinates and Area

How would we calculate an area using polar coordinates? Our basic increment of area will be shaped like a slice of pie. The slice of pie shown in Figure 1 has

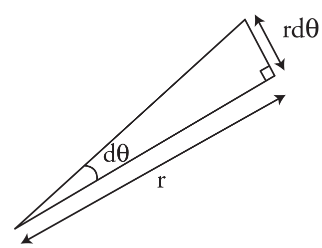

Figure 1: A slice of pie with radius r and angle $d\theta$.

a piece of a circular arc along its boundary with arc length $r\,d\theta$. We'll say that dA equals the area of the slice.

How do we express dA in terms of r and θ? The total area of the pie this was sliced from is πr^2. To find the area dA we note that the proportion of the total area covered equals the proportion of arc length covered. So:

$$\begin{aligned} \frac{dA}{\pi r^2} &= \frac{d\theta}{2\pi r} \\ dA &= \frac{r\,d\theta}{2\pi r} \cdot \pi r^2 \\ dA &= \frac{1}{2} r^2\,d\theta \end{aligned}$$

This is the basic formula for an increment of area in polar coordinates.

We want to use polar coordinates to compute areas of shapes other than circles. In this case r will be a function of θ. The distance between the curve and the origin changes depending on what angle our ray is at. Our center point of reference is the origin; we think of rays emerging from the origin at some angle θ; $r(\theta)$ is, roughly, the distance we must travel along that ray to get to the curve.

To find the area of a shape like this, we break it up into circular sectors with angle $\Delta\theta$. Since the curve is not a circle the circular sectors won't perfectly cover the region, so we just approximate the area of a wedge between the curve and the origin by:

$$\Delta A \approx \frac{1}{2} r^2 \Delta\theta.$$

If we take the limit as $\Delta\theta$ approaches zero our sum of sector areas will approach

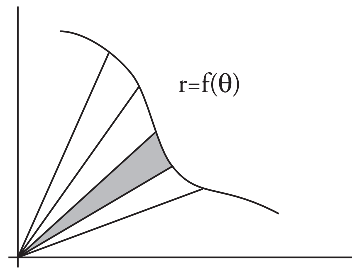

Figure 2: A slice from an oddly shaped pie.

the exact area and we get:

$$dA = \frac{1}{2}r^2\, d\theta.$$

This is very similar to letting Δx go to zero in a Riemann sum of rectangle areas.

In the limit, we have:

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\, d\theta.$$

Remember that we're assuming r is a function of θ.