Integral of $x^4 \cos x$

This problem provides a lot of practice with integration by parts.

Compute the integral of $x^4 \cos x$.

Solution

A single application of integration by parts simplifies, but does not solve, this integral. We must repeat integration by parts several times (or look up a reduction formula for integrating $x^n \cos x$) to complete the integration.

Taking the derivative of $\cos x$ does not simplify it, while the derivative of x^4 is slightly simpler. (In general, if we see sine, cosine or exponential functions, we should consider assigning them the role of v'.) We integrate by parts using the following assignments:

$$\begin{array}{ll} u = x^4 & v = \sin x \\ u' = 4x^3 & v' = \cos x \end{array}$$

to get:

$$\int x^4 \cos x \, dx = x^4 \sin x - \int 4x^3 \sin x \, dx.$$

We do not have a formula for $\int 4x^3 \sin x \, dx$, but a similar integration by parts will get us closer to one:

$$\begin{array}{ll} u = 4x^3 & v = -\cos x \\ u' = 12x^2 & v' = \sin x \end{array}$$

$$\Rightarrow \quad \int 4x^3 \sin x \, dx = -4x^3 \cos x + \int 12x^2 \cos x \, dx.$$

We must integrate by parts twice more before we can finish the problem.

$$\begin{array}{ll} u = 12x^2 & v = \sin x \\ u' = 24x & v' = \cos x \end{array}$$

$$\Rightarrow \quad \int 12x^2 \cos x \, dx = 12x^2 \sin x - \int 24x \sin x \, dx.$$

$$\begin{array}{ll} u = 24x & v = -\cos x \\ u' = 24 & v' = \sin x \end{array}$$

$$\Rightarrow \quad \int 24x \sin x \, dx = -24x \cos x + \int 24 \cos x \, dx.$$

Now we have all the pieces and can assemble them into our final answer:

$$\begin{aligned} \int x^4 \cos x \, dx &= x^4 \sin x - \int 4x^3 \sin x \, dx \\ &= x^4 \sin x - (-4x^3 \cos x + \int 12x^2 \cos x \, dx) \end{aligned}$$

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\begin{aligned}
&= x^4 \sin x + 4x^3 \cos x - (12x^2 \sin x - \int 24x \sin x \, dx) \\
&= x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x \\
&\quad +(-24x \cos x + \int 24 \cos x \, dx) \\
&= x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + 24 \sin x \\
&\quad +c.
\end{aligned}
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