Example: $f(x) = x^2$

Professor Jerison only does one simple example of computing the definite integral as a limit because computing integrals this way involves a lot of hard work.

In this example we'll use the first interesting curve, $f(x) = x^2$, with starting point $a = 0$. In order to see what the pattern is, we'll allow b to be arbitrary — we can replace it with a value after the calculation if we like.

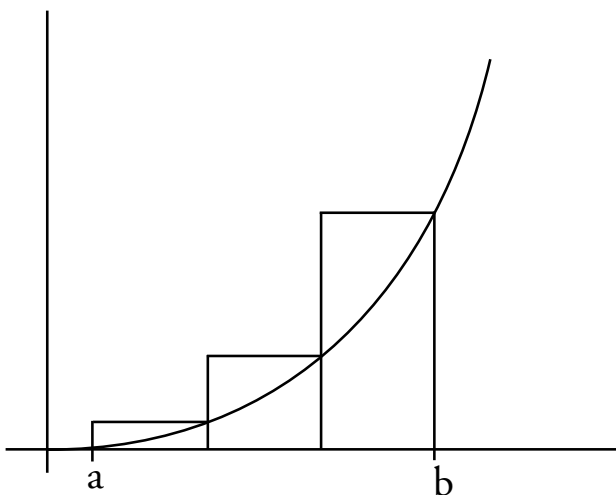We start by graphing $f(x)$ and identifying the region whose area we are computing.

Figure 1: $\int_a^b f(x)dx$ approximated by three rectangles

Next, we subdivide the interval from 0 to b into n pieces; in this class the pieces will always have equal length. An example with $n = 3$ is show in Figure 1. Each subdivision forms the base of a rectangle. The tops of the rectangles touch the graph; Professor Jerison has chosen to have the intersection of the rectangle and the graph in the upper right corner of the rectangle. Because of this, the sum of the areas of the rectangles will be slightly larger than the area under the curve.

Next, we find the areas of the rectangles. The nice thing about rectangles is that it's easy to compute their areas — just multiply the length of the base by the height. Since each of the n rectangle bases is the same size, the base of each rectangle has length $\dfrac{b}{n}$. The height of each rectangle will be given by $f(x_i) = x_i^2$, where x_i is the right endpoint of the base.

The first rectangle has base $\left[0, \frac{b}{n}\right]$, so its height is $\left(\frac{b}{n}\right)^2$ and its area is $\left(\frac{b}{n}\right)^3$. Information about other rectangles appears in the table below.

x (right endpoint of base)	$f(x)$ (height of rectangle)
$\frac{b}{n}$	$\left(\frac{b}{n}\right)^2$
$\frac{2b}{n}$	$\left(\frac{2b}{n}\right)^2$
$\frac{3b}{n}$	$\left(\frac{3b}{n}\right)^2$
$\vdots$	$\vdots$
$\frac{nb}{n}$	b^2

Now we can easily find the areas of the rectangles; then we add them up to get an approximation of the area under the curve:

$$\underbrace{\left(\frac{b}{n}\right)}_{\text{base}}\underbrace{\left(\frac{b}{n}\right)^2}_{\text{height}} + \left(\frac{b}{n}\right)\left(\frac{2b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{3b}{n}\right)^2 + \cdots + \left(\frac{b}{n}\right)\left(\frac{nb}{n}\right)^2$$

This is complicated. We'll start by simplifying it and work toward evaluating the limit as n goes to infinity (i.e. when the base of the rectangle is infinitesimal.) It turns out that it's easier to calculate the limit than to calculate the sum of the areas of the rectangles; that's why we study calculus.

First, we can factor out $\left(\frac{b}{n}\right)^3$:

$$\left(\frac{b}{n}\right)\left(\frac{b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{2b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{3b}{n}\right)^2 + \cdots + \left(\frac{b}{n}\right)\left(\frac{nb}{n}\right)^2 =$$

$$\left(\frac{b}{n}\right)\frac{b^2}{n^2} + \left(\frac{b}{n}\right)\frac{2^2b^2}{n^2} + \left(\frac{b}{n}\right)\frac{3^2b^2}{n^2} + \cdots + \left(\frac{b}{n}\right)\frac{n^2b^2}{n^2} =$$

$$\frac{b^3}{n^3}(1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2)$$

We want to take the limit as n goes to infinity. What makes this difficult is the sum $1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2$. We're going to use a geometric "trick" to draw a picture representing this quantity.

Geometric Calculation of Sum of Squares:

Imagine you're building a pyramid. The base of the pyramid is an n by n square of n^2 cubes. The next layer is $n-1$ by $n-1$, and has $(n-1)^2$ blocks in it. The top view of the pyramid shows a series of concentric squares. In profile, the pyramid looks like a triangle formed out of rectangles with height 1 and length n, $(n-1)$, etc. The left and right sides of that triangle have slopes 2 and -2.

The volume of this pyramid is $n^2 + (n-1)^2 + \ldots + 3^2 + 2^2 + 1^2$, which equals the difficult sum from our definite integral. This volue is slightly larger than that of the ordinary pyramid with base n and height n which is the largest ordinary pyramid contained entirely inside our stair-step pyramid. We know that the volume of that inside pyramid is $\frac{1}{3} \cdot$ base $\cdot$ height, or $\frac{1}{3}n^2 \cdot n$. So

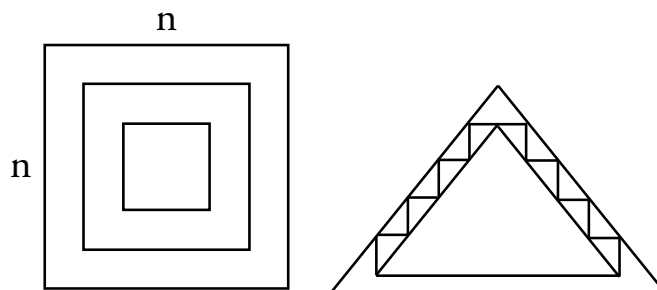$$\frac{1}{3}n^3 < 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2.$$

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Figure 2: Top and side views of a stair-step pyramid.

We've compared our difficult sum, which equaled the volume of a stair-step pyramid, to the volume $\frac{1}{3}n^3$ of the biggest ordinary pyramid that fit inside it. This is a step toward replacing the difficult sum by something much simpler.

Next we compare the volume of the stair-step pyramid to the volume of the smallest ordinary pyramid that can contain it. That pyramid has base $n+1$ and height $n+1$, and so has volume $\frac{1}{3}(n+1)^3$. Hence:

$$\frac{1}{3}n^3 < 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 < \frac{1}{3}(n+1)^3.$$

We now divide everything by n^3 to get:

$$\frac{1}{3} < \frac{1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2}{n^3} < \frac{1}{3}\frac{(n+1)^3}{n^3} = \frac{1}{3}\left(1 + \frac{1}{n}\right)^3.$$

If we let n go to infinity, we find that the left and right sides of this inequality approach $\frac{1}{3}$ and so the center expression

$$\frac{1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2}{n^3} \longrightarrow \frac{1}{3} \text{ as } n \longrightarrow \infty.$$

The sum of the areas of the rectangles under the graph of x^2 was

$$\frac{b^3}{n^3}(1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2).$$

As n approaches infinity, this area approaches $\frac{b^3}{3}$. So the total area between the graph of $f(x) = x^2$ and the interval $[0, b]$ is:

$$\int_0^b x^2 dx = \frac{1}{3}b^3.$$

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Question: Why did we leave the $\left(\frac{b}{n}\right)^3$ out for this step?

Answer: Part of the answer is that we know what we're heading for. We understand the quantity $\left(\frac{b}{n}\right)^3$. However, the difficult sum $(1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2)$ is growing larger and larger in a way we don't entirely understand. So we separate out the difficult sum and concentrate on that. When we do, we discover that it's very, very similar to n^3, and is even more similar to $\frac{1}{3}n^3$. Once we understand that we can use it in the original equation, cancel the n^3's, and get our result.

This is what you always do if you analyze these kinds of sum; you factor out whatever you understand and end up with a sum like this. You should expect this to happen every time you're faced with such a sum.

In summary, the steps we followed to find the area under the curve were:

1. Graph the function

2. Subdivide into n intervals of length $\frac{b-a}{n}$

3. Compute the heights $f\left(\frac{i(b-a)}{n}\right)$ of the rectangles

4. Compute the areas of the rectangles

5. Sum the areas of the rectangles

6. Find the limit of this sum as n goes to infinity.

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18.01SC Single Variable Calculus
Fall 2010