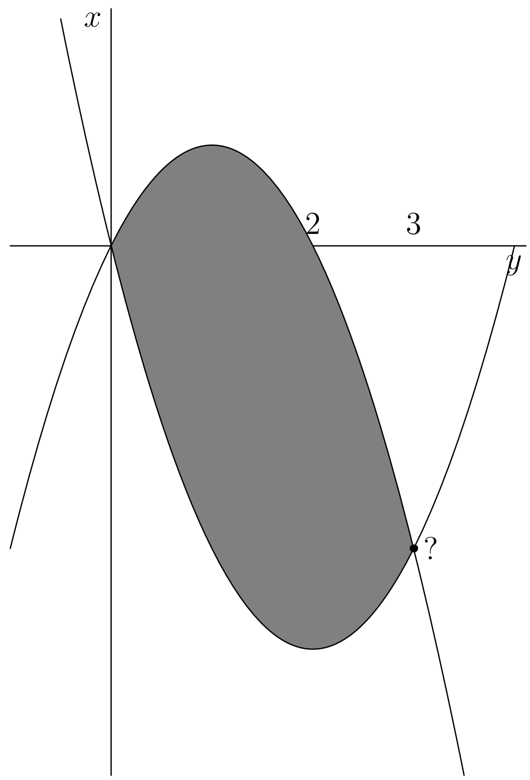
1. Compute the area between the curves $x = y^2 - 4y$ and $x = 2y - y^2$.

Let $f(y) = y^2 - 4y = y(y-4)$. $f(y) = 0$ when $y = 0$ or $y = 4$.

Let $g(y) = 2y - y^2 = y(2-y)$. $g(y) = 0$ when $y = 0$ or $y = 2$.



The graphs of f and g intersect at $(0,0)$ and one other point. Find that point:

$$\begin{aligned} f(y) &= g(y) \\ y^2 - 4y &= 2y - y^2 \\ 2y^2 - 6y &= 0 \\ 2y(y-3) &= 0 \end{aligned}$$

The graphs intersect at $y = 0$ and at $y = 3$. When $y = 3$, $f(y) = -3$ so the second point of intersection is $(3, -3)$. (Check this by finding $g(3)$.)

Over the interval between intersections of the graphs, $g(y) > f(y)$. The distance between graphs is:

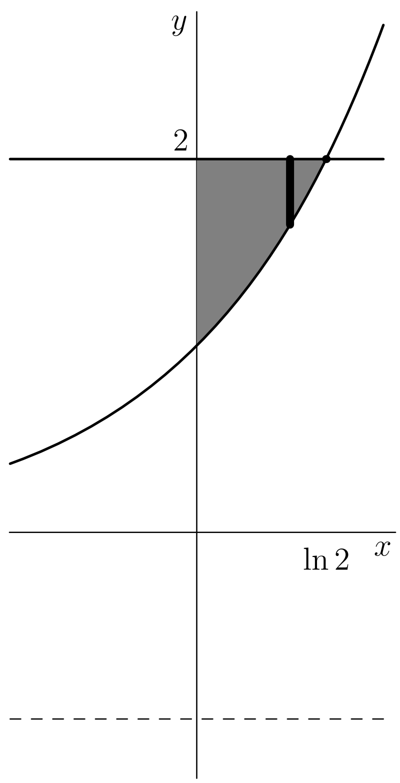
$$g(y) - f(y) = (2y - y^2) - (y^2 - 4y) = 6y - 2y^2.$$

The area between graphs is:

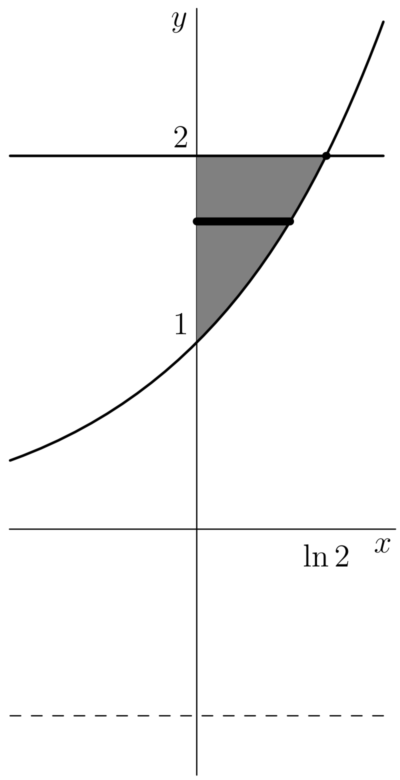
$$\int_0^3 6y - 2y^2\, dy = \left[3y^2 - \frac{2}{3}y^3\right]_0^3$$

$$
\begin{aligned}
&= \left(3 \cdot 3^2 - \frac{2}{3} \cdot 3^3\right) - 0 \\
&= 27 - 18 \\
&= 9.
\end{aligned}
$$

2. Find the volume of the solid obtained by revolving the region bounded by the curves $y = e^x$, $y = 2$, and $x = 0$ about the line $y = -1$. You only need to give a definite integral expressing the volume. Do not solve the integral.



Washer method: $\displaystyle\int_0^{\ln 2} \pi(3^2 - (1 + e^x)^2)\, dx$



Shell method: $\int_1^2 2\pi(y+1)\ln y\, dy$

3. Evaluate each of the following expressions

   (a)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + i \cdot \frac{3}{n}\right)^2 \frac{3}{n}$$

Strategy: interpret this as a Riemann sum and find its value by integrating.
Consider the interval $[0,3]$ cut into n parts.
Consider the function $f(x) = (1+x)^2$.
The right Riemann Sum is:

$$\sum_{i=1}^{n} \left(1 + i\frac{3}{n}\right)^2 \frac{3}{n}$$

$$\text{So } \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + i\frac{3}{n}\right)^2 \frac{3}{n} = \int_0^3 (1+x)^2\, dx$$

$$= \int_0^3 1 + 2x + x^2\, dx$$

$$
\begin{aligned}
&= \left[x + x^2 + \frac{1}{3}x^3\right]_0^3 \\
&= (3+9+9) - 0 \\
&= 21.
\end{aligned}
$$

(b) The value $f(4)$ for the continuous function f satisfying

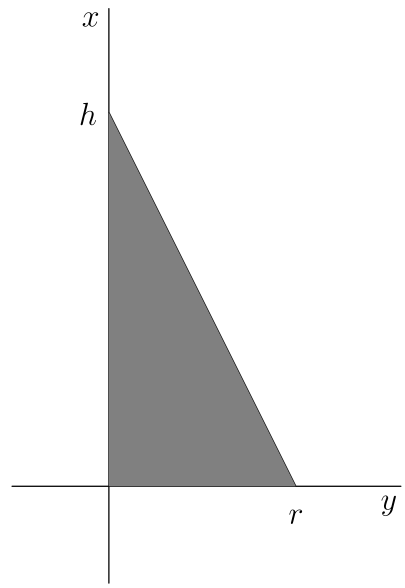
$$x \sin \pi x = \int_0^{x^2} f(t)\, dt$$

Strategy: apply the fundamental theorem of calculus.

$$
\begin{aligned}
\frac{d}{dx}(x \sin \pi x) &= \frac{d}{dx} \int_0^{x^2} f(t)\, dt \\
\Rightarrow \sin \pi x + \pi x \cos \pi x &= f(x^2) \cdot 2x \\
\Rightarrow f(x^2) &= \frac{1}{2x} \sin \pi x + \frac{1}{2x} \pi x \cos \pi x \\
\Rightarrow f(4) = f(2^2) &= \frac{1}{2 \cdot 2} \sin 2\pi + \frac{\pi}{2} \cos 2\pi \\
&= \frac{\pi}{2}
\end{aligned}
$$

4. (a) Find the centroid (i.e. center of mass) of a right triangle with height h and base r (assuming the triangle has uniform density). For a plane figure with uniform density, the coordinates of the center of mass are given by weighted averages, where the weighting function is the moment of inertia:

$$\left(\frac{\int x f(x)\, dx}{\int f(x)\, dx}, \frac{\int y g(y)\, dy}{\int g(y)\, dy} \right).$$



Note that the hypotenuse of the triangle lies on a line with equation $y = h - \frac{h}{r}x$.

You may know from the homework that the center of mass lies at the centroid $(h/3, r/3)$.

If not, you will need to calculate the x and y coordinates of the center of mass separately. The formula for the x coordinate of the center of mass looks something like:

$$\frac{\int xf(x)\,dx}{\int f(x)\,dx}.$$

In this case, the numerator of this expression is:

$$\begin{aligned}
\int_0^r x\left(h - \frac{h}{r}x\right)\,dx &= \int_0^r hx - \frac{h}{r}x^2\,dx \\
&= \left[\frac{h}{2}x^2 - \frac{h}{3r}x^3\right]_0^r \\
&= \left(\frac{h}{2}r^2 - \frac{h}{3r}r^3\right) - 0 \\
&= \frac{h}{6}r^2
\end{aligned}$$

The denominator is just the area of the triangle: $\frac{1}{2}rh$. So the x coordinate of the center of mass is:

$$\frac{hr^2/6}{hr/2} = \frac{r}{3}.$$

For the y coordinate, we note that the hypotenuse lies on the line with equation $x = r - \frac{r}{h}y$ and so the numerator will be:

$$\begin{aligned} \int_0^h y\left(r - \frac{r}{h}y\right)\,dy &= \int_0^h ry - \frac{r}{h}y^2\,dy \\ &= \left[\frac{r}{2}y^2 - \frac{r}{3h}y^3\right]_0^h \\ &= \left(\frac{r}{2}h^2 - \frac{r}{3h}h^3\right) - 0 \\ &= \frac{r}{6}h^2 \end{aligned}$$

Dividing by the area of the triangle, we find that the y coordinate of the center of mass is:

$$\frac{rh^2/6}{hr/2} = \frac{h}{3}.$$

The centroid of the right triangle with height h and base r shown in the figure above lies at $\left(\frac{r}{3}, \frac{h}{3}\right)$.

(b) Pappus' Theorem says that the volume of the solid formed by rotating a region is the area of the region times the distance traveled by the rotating centroid. Use Pappus' Theorem and your answer in the previous part to find the volume of a cone with height h and base radius r.

We can form a cone with height h and base radius r by rotating the triangle above about the y axis. The area of the rotated region is $\frac{1}{2}rh$. The centroid lies distance $r/3$ from the y axis, so it travels a distance of $2\pi r/3$ as it is rotated.

Hence, by Pappus' Theorem, the volume of the cone is:

$$\frac{1}{2}rh \cdot 2\pi\frac{r}{3} = \frac{\pi r^2 h}{3}.$$

5. Given a definite integral

$$\int_a^b f(x)\,dx,$$

let T_n be the *trapezoid* approximation with n intervals, M_n the *midpoint* approximation using n intervals, and S_{2n} the *Simpson's rule* approximation using $2n$ intervals. Prove that

$$\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}.$$

We divide the interval $[a, b]$ into $2n$ intervals.

Let $x_0 = a,\ x_{2n} = b,\ x_i = a + \dfrac{(b-a)i}{2n}$.

Then:

$$\begin{aligned}
T_n &= \frac{b-a}{2n}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{n-1} f(x_{2i})\right) \\
M_n &= \frac{b-a}{n}\left(\sum_{i=1}^{n} f(x_{2i-1})\right) \\
S_{2n} &= \frac{b-a}{6n}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{2n-1}) + x_{2n}\right) \\
&= \frac{b-a}{6n}\left(f(x_0) + f(x_{2n}) + 4\sum_{i=1}^{n} f(x_{2i-1}) + 2\sum_{i=1}^{n-1} f(x_{2i})\right)
\end{aligned}$$

Therefore:

$$\begin{aligned}
\frac{1}{3}T_n + \frac{2}{3}M_n &= \frac{1}{3}T_n + \frac{4}{6}M_n \\
&= \frac{b-a}{6n}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{n-1} f(x_{2i}) + 4\sum_{i=1}^{n} f(x_{2i-1})\right) \\
&= S_{2n}
\end{aligned}$$

6. A tank contains 1000 L of brine (that is, salt water) with 15 kg of dissolved salt. Pure water enters the top of the tank at a constant rate of 10 L / min. The solution is thoroughly mixed and drains from the bottom of the tank at the same rate so that the volume of liquid in the tank is constant.

   (a) Find a differential equation expressing the rate at which salt leaves the tank.

   Let $s(t)$ = amount of salt in kg. at time t. Then

   $$\frac{ds}{dt} = -10 \text{ L/min} \cdot \frac{s(t)}{1000}\frac{\text{kg}}{\text{L}} = -\frac{s(t)}{100} \text{ kg/min.}$$

   (b) Solve this differential equation to find an expression for the amount of salt (in kg) in the mixture at time t.

   Use separation of variables:

   $$\frac{ds}{s(t)} = -\frac{1}{100}\,dt.$$

Then integrate:

$$\ln(s(t)) = -\frac{1}{100}t + c.$$

To get rid of the logarithm, exponentiate both sides, letting $k = e^c$:

$$\begin{aligned}
\ln(s(t)) &= -\frac{1}{100}t + c \\
e^{\ln(s(t))} &= e^{-\frac{1}{100}t+c} \\
s(t) &= e^{-\frac{1}{100}t}e^c \\
s(t) &= ke^{-\frac{1}{100}t}
\end{aligned}$$

We know $s(0) = 15$, so $k = 15$. Hence:

$$\boxed{s(t) = 15e^{-\frac{1}{100}t}.}$$

(c) How long does it take for the total amount of salt in the brine to be reduced by half its original amount? (Recall $\ln 2 \approx .693$.)

We need:

$$\begin{aligned}
e^{-\frac{1}{100}t} &= \frac{1}{2} \\
\ln(e^{-\frac{1}{100}t}) &= \ln(\frac{1}{2}) \\
-\frac{1}{100}t &= -\ln 2 \\
t &= 100 \cdot \ln 2 \approx \boxed{\text{69.3 minutes.}}
\end{aligned}$$