Differential Equations and Slope, Part 1

Suppose the tangent line to a curve at each point (x, y) on the curve is twice as steep as the ray from the origin to that point. Find a general equation for this curve. (See Fig. 1.)

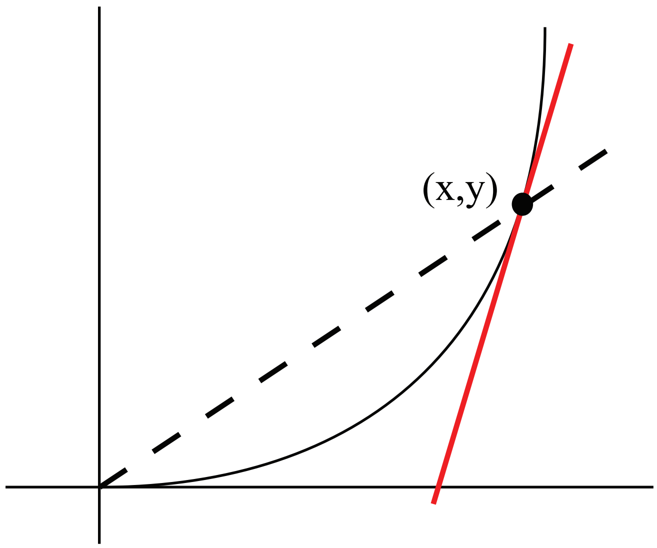

Figure 1: The slope of the tangent line (red) is twice the slope of the ray from the origin to the point (x, y).

This type of problem can be described very succinctly using differential equations. The slope of the tangent line is $\frac{dy}{dx}$. The slope of the ray from $(0, 0)$ to (x, y) is $\frac{y}{x}$. Since the slope of that ray is twice the slope of that ray, we get the differential equation:

$$\frac{dy}{dx} = 2\left(\frac{y}{x}\right).$$

We only have one method for solving differential equations; use it.

$$\begin{aligned}
\frac{dy}{dx} &= 2\frac{y}{x} \\
\frac{dy}{y} &= \frac{2\,dx}{x} \quad \text{(separate variables)} \\
\int \frac{dy}{y} &= \int \frac{2}{x}\,dx \quad \text{(integrate both sides)} \\
\ln|y| &= 2\ln|x| + c \quad \text{(antidifferentiate)} \\
e^{\ln|y|} &= e^{2\ln|x|+c} \quad \text{(apply an inverse function to isolate y)} \\
e^{\ln|y|} &= e^c e^{2\ln|x|} \quad \text{(exponentiate)}
\end{aligned}$$

$$\begin{aligned} e^{\ln|y|} &= e^c(e^{\ln|x|})^2 \\ |y| &= e^c x^2 \quad (e^{2\ln|x|} = x^2\ ) \end{aligned}$$

There is an absolute value in this solution. When $y > 0$ we get $y = e^c x^2$. When $y < 0$ we get $y = -e^c x^2$. Based on prior experience we guess that the solution will be $y = ax^2$, where $a = \pm e^c$ or $a = 0$.

Because we divided by y in our calculations our solution doesn't include the case in which $a = 0$ and $y = 0x^2$. Graph the equation $y = 0$ and confirm that at each point on the graph the slope of the tangent line is twice the slope of the ray joining that point to the origin; this confirms that $y = 0x^2$ is a solution.

We conclude that the general solution to the problem is:

$$y = ax^2$$

where a could be positive, negative or zero. Some possible solutions include:

$$\begin{aligned} y &= x^2 \quad (a = 1) \\ y &= 2x^2 \quad (a = 2) \\ y &= -x^2 \quad (a = -1) \\ y &= 0x^2 = 0 \quad (a = 0) \\ y &= -2y^2 \quad (a = -2) \\ y &= 100x^2 \quad (a = 100) \end{aligned}$$

Some representatives of this family of curves are shown in black in Fig. 2.

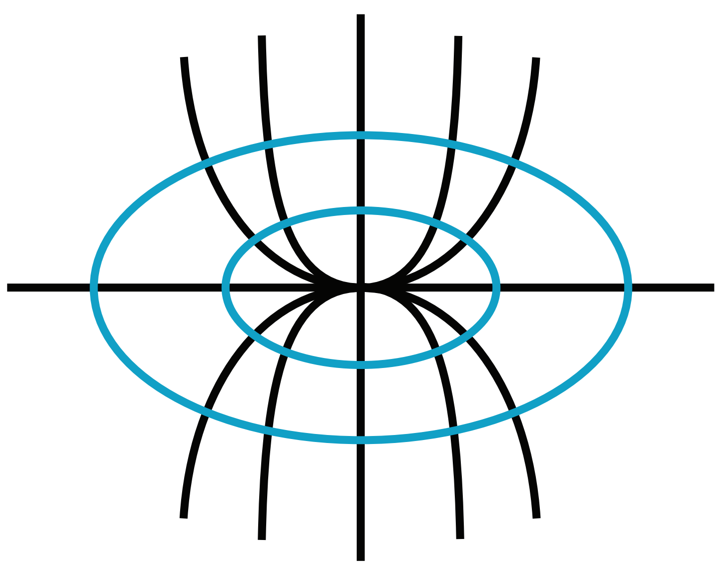

Figure 2: Parabolic curves, shown in black.

If we want to check our work, we can do so by taking the derivative:

$$\begin{aligned} y &= ax^2 \\ \frac{dy}{dx} &= 2ax \end{aligned}$$

Since $2ax = \frac{2ax^2}{x}$, we have $\frac{dy}{dx} = \frac{2y}{x}$. This works for $a > 0$, $a < 0$ and $a = 0$, so this solution is valid for all those values of a.

Warning: Notice that in the equation $\frac{dy}{dx} = \frac{2y}{x}$, $\frac{2y}{x}$ is undefined at $x = 0$. As you can see from Fig. 2, knowing the value of the function and its derivative at $x = 0$ doesn't tell us how the function will behave elsewhere. This is bad — for one thing, it contradicts our understanding of linear approximation.

What goes wrong is that the rate of change is not specified when $x = 0$. If you think carefully about what this function is doing, it could follow one branch when $x < 0$ and a completely different branch when $x > 0$. That's a very subtle point; you won't be asked to discuss this problem in your homework, but you should be aware that when x is equal to zero there's a problem for this differential equation.

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18.01SC Single Variable Calculus
Fall 2010

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