Newton's Method

Today we'll discuss the accuracy of Newton's Method.

Recall how Newton's method works: to find the point at which a graph crosses the x-axis you make an initial guess x_0 at the x-coordinate of that crossing. You then find the tangent line to the graph at x_0 and use it to improve your guess: x_1 is the x-coordinate at which the tangent line crosses the x-axis. (See Fig. 1.) You can now draw the tangent line at x_1 to get a new guess x_2, and so on.

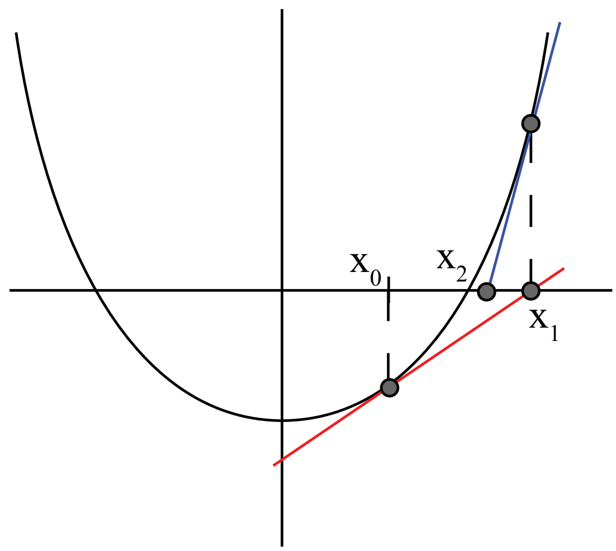

Figure 1: Illustration of Newton's Method

In algebraic terms,

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

Figure 2 illustrates the k^{th} iteration of Newton's method.

If we're going to use this to get numerical approximations of solutions, we should know how accurate it is. If x is the exact value of the solution, then x_1 is $E_1 = |x - x_1|$ away from the exact answer. The error in our approximation at step n is $E_n = |x - x_n|$.

Last time we saw that error values of $E_n = |\sqrt{5} - x_n|$ quickly became very close to zero. It turns out that $E_2 \sim E_1^2$. So if $E_0 = 10^-1$, the size of the error can be expected to decrease as follows:

E_0	E_1	E_2	E_3	E_4
10^{-1}	10^{-2}	10^{-4}	10^{-8}	10^{-16}

The number of digits of accuracy doubles at each step!

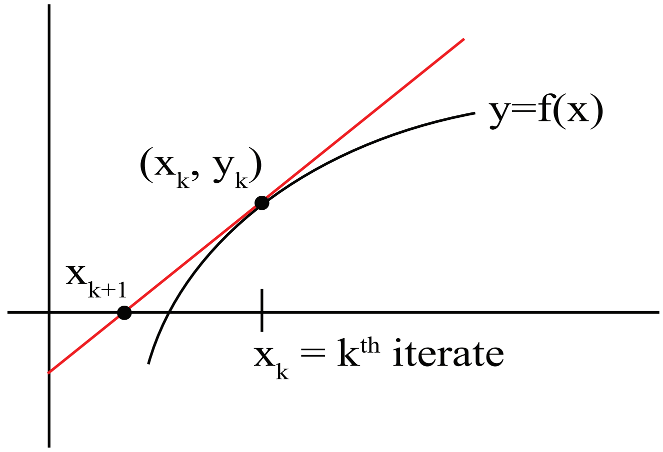

Figure 2: Illustration of Newton's Method.

Newton's method works (very) well if $|f'|$ is not too small, $|f''|$ is not too big, and x_0 starts near the solution x.