Linear Approximation and the Definition of the Derivative

Another way to understand the formula for linear approximation involves the definition of the derivative:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

Look at this backward:

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(x_0)$$

We can interpret this to mean that:

$$\frac{\Delta f}{\Delta x} \approx f'(x_0) \quad \text{when } \Delta x \approx 0.$$

In other words, the average rate of change $\frac{\Delta f}{\Delta x}$ is nearly the same as the infinitesimal rate of change $f'(x_0)$.

We can see that this is the same as our original formula

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

if we multiply both sides by Δx and remind ourselves what Δx and Δf are abbreviations for:

$$\begin{aligned}
\frac{\Delta f}{\Delta x} &\approx f'(x_0) \\
\Delta x \cdot \frac{\Delta f}{\Delta x} &\approx f'(x_0) \cdot \Delta x \\
\Delta f &\approx f'(x_0) \cdot \Delta x \\
f(x) - f(x_0) &\approx f'(x_0)(x - x_0) \\
f(x) &\approx f(x_0) + f'(x_0)(x - x_0)
\end{aligned}$$

So we have two different ways of writing a formula for linear approximation. When you're solving linear approximation problems, try to choose the most appropriate formula for the problem you're working on.