Definition of e

Recall that:

$$M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}.$$

is the value for which $\frac{d}{dx}a^x = M(a)a^x$, the value of the derivative of a^x when $x = 0$, and the slope of the graph of $y = a^x$ at $x = 0$. We need to know what $M(a)$ is in order to find out what the derivative of a^x is. It turns out that the easiest way to understand $M(a)$ is to give up trying to calculate it and to *define* e as the number such that $M(e) = 1$.

Leaving aside the question of whether such a number e exists, let's discuss what such a number would do for us. Since $M(e) = 1$,

$$\frac{d}{dx}e^x = e^x.$$

This is an incredibly important formula and is the only thing we've said so far this lecture that you need to memorize. Also, the slope of the tangent line to $y = e^x$ at $x = 0$ has slope 1. You can confirm this by plugging $x = 0$ into $\frac{d}{dx}e^x = e^x$.

But we still don't know what e is, or even if there is such a number. How do we know that there is *any* number a for which the slope of the tangent line to $y = a^x$ is 1 when $x = 0$?

First notice that as the base a increases, the graph of $y = a^x$ gets steeper. Is the slope ever 1?

If $a = 1$, $a^x = 1$ for all x and the slope of the tangent line to the (very simple) graph at $x = 0$ is 0. Although we may not be able to compute the slope exactly, we can use secant lines to estimate the slope $M(a)$ for $a = 2$ and $a = 4$ geometrically. Look at the graph of 2^x in Fig. 1. The secant line from $(0, 1)$ to $(1, 2)$ of the graph $y = 2^x$ has slope 1. We can see from the picture that the slope of $y = 2^x$ at $x = 0$ is less than the slope of this secant line: $M(2) < 1$ (see Fig. 1).

Next, look at the graph of 4^x in Fig. 2. The secant line from $(-\frac{1}{2}, \frac{1}{2})$ to $(1, 0)$ on the graph of $y = 4^x$ has slope 1. We see that the slope of $y = 4^x$ at $x = 0$ is greater than the slope of the secant, so $M(4) > 1$ (see Fig. 2).

Assuming our function M is continuous, we conclude that somewhere in between 2 and 4 there is a base whose slope at $x = 0$ is 1.

Thus we can *define* e to be the unique number such that

$$M(e) = 1$$

or, to put it another way,

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

or, to put it yet another way,

$$\frac{d}{dx}(e^x) = 1 \quad \text{at } x = 0$$

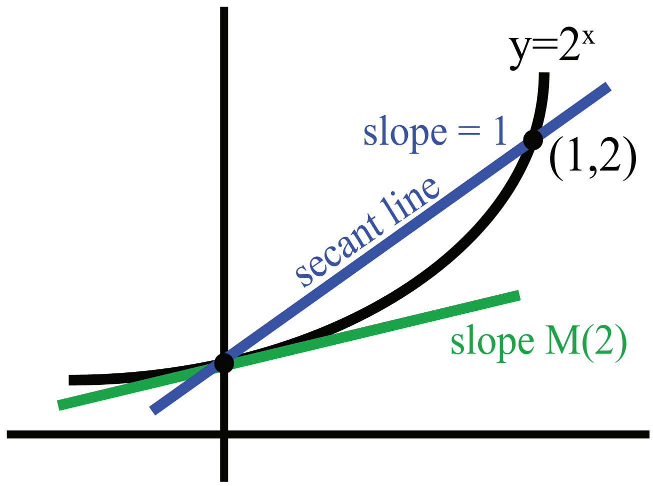

Figure 1: Slope $M(2) < 1$

Another way to convince ourselves that e must exist is to start with the graph of $f(x) = 2^x$ (recalling that $M(2) < 1$) and think about the function $f(kx) = 2^{kx}$. As k increases, the graph of $y = f(kx)$ is compressed horizontally and the slope of the tangent line to the graph of $y = f(x)$ continuously grows steeper. So, for some value of k between 1 and 2, the slope of that tangent line must be 1. So e exists and is between $2^1 = 2$ and $2^2 = 4$.

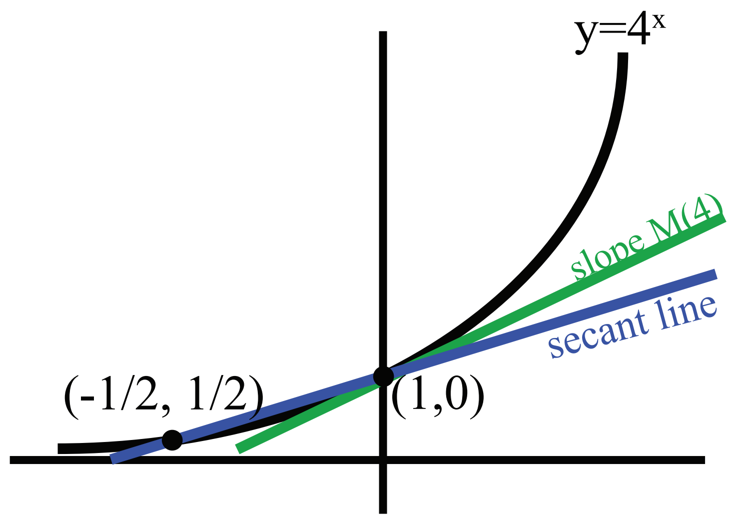

Figure 2: Slope $M(4) > 1$