Questions and Answers

It's difficult to visualize the relationship between the arc length θ and the segment length $1 - \cos(\theta)$ in the geometric argument that:

$$\lim_{x \to 0} \frac{1 - \cos(\theta)}{\theta} = 0.$$

Professor Jerison spends ten minutes answering student questions and clarifying his argument.

Doesn't θ also tend to 0?

Yes. Whenever we take a derivative we're dividing by a quantity that tends to zero. The quantity $\lim_{x \to 0} \frac{1 - \cos(\theta)}{\theta}$ happens to be the derivative of $\cos \theta$ evaluated at zero. It should not surprise us that this ratio tends to zero divided by zero. What we're interested in is that as θ goes to zero the straight line approximating the arc with length θ becomes much longer than the "gap" of length $1 - \cos \theta$.

As the angle θ decreases and we "zoom in", it turns out that the ratio between the size of the "gap" and the length of the "bowstring" gets smaller and smaller – the gap shrinks faster than the bowstring. So, as the measure of angle θ approaches 0, the ratio $\frac{1 - \cos(\theta)}{\theta}$ also approaches 0.

In the example of $\dfrac{\sin \theta}{\theta}$ we were comparing two quantities that were about the same size and which approached zero at approximately the same rate, so their ratio approached 1.

Where is the Stata Center?

The Stata Center is a building at MIT that is approximately 100 meters away from the hall in which this lecture is given.

Is $\frac{\cos \theta - 1}{\theta}$ the same as $\frac{1 - \cos(\theta)}{\theta}$?

$\dfrac{\cos \theta - 1}{\theta}$ is the negative of $\dfrac{1 - \cos(\theta)}{\theta}$.

$$-\frac{1 - \cos(\theta)}{\theta} = \frac{-1 + \cos \theta}{\theta} = \frac{\cos \theta - 1}{\theta}$$

What do you mean by "arc length"?

The arc in question is the portion of the unit circle swept out by the angle θ. Its arc length is its length.

Why is its length θ?

Because we are measuring the angle θ in radians and because the radius of the circle has length 1, the length of the arc swept out by θ will equal the angle θ.

This is especially important in the discussion of $\frac{\sin\theta}{\theta}$. If we were measuring θ in degrees, for example, we would have to compute the arc length before we could compare it to the vertical distance $\sin\theta$. The arc length only equals θ when θ is measured in radians.

The best way to measure θ in this calculation is by length along the unit circle, which is what radians are. Unfortunately, we won't see a proof of this until near the end of the course.

Why is the length of the radius equal to 1?

When working with trig functions like $\cos(x)$ it's extremely useful to think about a circle with radius 1 centered at the point $(0,0)$ because a point (x, y) at angle θ on the circle will have coordinates $(\cos\theta, \sin\theta)$.

The radius shown on the blackboard looks like it's more than one unit long because we "zoomed in" on it as θ became too small to see.

I'm having trouble visualizing how $\frac{1-\cos(\theta)}{\theta} \to 0$.

You are probably not the only one.

First, visualize angle θ getting smaller and smaller.

When θ is very, very small, we can't see what's going on unless we "blow up" or "zoom in on" the picture. (In fact, if we let θ decrease all the way to 0, the triangle collapses and $\frac{1-\cos(\theta)}{\theta} = \frac{0}{0}$. That's why we have to use a limit.)

To estimate the value of $\frac{1-\cos(\theta)}{\theta}$ as θ shrinks, we need to look at geometric interpretations of both the numerator $(1 - \cos\theta)$ and the denominator θ and compare them.

The numerator is the length of the tiny "gap" and the denominator is half the length of the "bow string". From the picture, we see that the gap is much shorter than the bow string when θ is small.

As θ shrinks, we zoom in further, the bow of the circular arc gets closer and closer to the bow string, and so the gap gets smaller and smaller while the bow string appears to stay the same length. So the ratio of the size of the gap to the length of the bow string gets smaller and smaller, approaching zero.

Not only does the size of the gap go to zero, it goes to zero faster than the length of the bow string does.

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