Limits and Discontinuity

For which of the following should one use a one-sided limit? In each case, evaluate the one- or two-sided limit.

1. $\lim_{x \to 0} \sqrt{x}$

2. $\lim_{x \to -1} \frac{1}{x+1}$

3. $\lim_{x \to 1} \frac{1}{(x-1)^4}$

4. $\lim_{x \to 0} |\sin x|$

5. $\lim_{x \to 0} \frac{|x|}{x}$

Solutions

1. $\lim_{x \to 0} \sqrt{x}$

   The function $f(x) = \sqrt{x}$ is only defined for positive values of x, so we must use the one sided right hand limit here. Some work with a calculator or a graph quickly reveals that $\lim_{x \to 0^+} \sqrt{x} = 0$.

2. $\lim_{x \to -1} \frac{1}{x+1}$

   The value of $\frac{1}{x+1}$ increases without bound as x approaches -1 and the value of the denominator approaches 0. For $x < -1$ the expression takes on large, negative values and when $x > -1$ its value is large and positive so the left hand limit differs from the right hand limit and we are forced to use one-sided limits:

   $$\lim_{x \to -1^-} \frac{1}{x+1} = -\infty \qquad \lim_{x \to -1^+} \frac{1}{x+1} = \infty.$$

3. $\lim_{x \to 1} \frac{1}{(x-1)^4}$

   Once again the value of the expression increases without bound, but in this case its value is always positive. We can say that $\lim\limits_{x \to 1} \frac{1}{(x-1)^4} = \infty$.

4. $\lim_{x \to 0} |\sin x|$

   Here,

   $$\lim_{x \to 0^+} |\sin x| = \lim_{x \to 0^-} |\sin x| = 0$$

   so there is no need to use the one-sided limit.

5. $\lim_{x \to 0} \frac{|x|}{x}$

A one-sided limit is necessary because:

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1.$$