

CHRISTINE

Welcome back to recitation.

BREINER:

Today we're going to talk about something you've been seeing in the lectures. Specifically, we're going to talk about continuity and differentiability. And we're going to use an example to see a little bit what the difference is. That how you can be continuous and not necessarily differentiable. And how it's a little stronger to have differentiability.

So we're going to deal with a piecewise function. I'm going to ask a question, I'll give you a little bit of time to work on it, and then we'll come back. So the question is the following: for what values of a and b is the following function either first, continuous, or second, differentiable? So the function is defined in the following way: f of x is going to be equal to the function x squared plus 1 when x is bigger than 1. And it's going to be equal to a linear function where you get to pick the a , which is the slope, and you get to pick the b , which is a y -intercept, if x is less than or equal to 1.

So what I've drawn so far, so that you can see, is that I've drawn the part of x squared plus 1 if x is bigger than 1. So I've taken the x squared function that starts at $(0, 0)$ -- or the vertex is at $(0, 0)$ -- I've shifted it up one unit, and then I've chopped off everything left of and including the value at x equal 1.

So my question, again, is, what choices do I have for a and b to either first, figure out what choices for a and b allow this function to be continuous when I put in the left part-- so the values when x is less than or equal to 1-- and what values for a and b allow this function to be differentiable?

So I'm going to give you a moment to work on it yourself, and then we'll come back and I'll work through them for you.

OK. So what we were doing, again, is we were trying to answer this question, choose values for a and b that make-- first, let's look at the continuity question.

OK. So what we really need is, we need to have a linear function that ultimately goes through this point, whatever this point is. We have to figure out what that point is, and then we can figure out what values of a and b will allow that to work.

So if I want a continuous function, these a and b , again, they have to be such that the line goes straight through that point. So what is that point? Well, the x -value is 1. So what would the y value have to be in order to fill in that circle? We can actually look at f of x and we can say, well, if I wanted it to be continuous, then the y -value I need here, I need at x equal 1, is going to be whatever the y -value is here at x equal 1.

So let me write that. Make a little space over here. So to answer question one, I need a and b so that at x equal 1, y is equal to-- well, if I put in 1 here, what do I get? I get 1 squared plus 1. So I get 2. So I need y to equal 2 when x is equal to 1. OK?

So, in other words, down here-- again, let me just say before I go on, this point is 1 comma 2. So what I need is a line that goes through the point 1 comma 2. OK? So what is that? That would be, in this case, that would be, the output is 2, and the input is a times-- well, x is 1-- 1 plus b . So that means a plus b has to equal 2.

And that actually represents all the solutions. So there are an infinite number solutions I could have. And if you think about it, I could draw some of these, I could draw some of these lines.

So let's take, for instance, b equals 2 and a equals 0. When b is 2 and a is 0, it's the constant, it's the constant function f of x equals 2. It's a straight line. That would graph-- going to graph that here. That would graph straight across, and that would fill in right here, and this would be if we had on this side, this is f of x equals 2. OK?

I could also have chosen a plus b equals 2. I could also have chosen a equals 1 and b equals 1. So I could have slope 1 and intercept 1.

Notice, by the way, this one is not differentiable. Right? It's obviously-- there's a significant break there. We'll see also, with a equal 1 and b equals 1, it's also not differentiable. So a equals 1 is slope 1, and b equals 1 is intercept is 1. That's this line. So this is kind of running out of room, but f of x equals x plus 1. Hopefully you can see that. f of x equals x plus 1. f of x equals 2.

This one looks a little bit closer to being almost like the tangent lines are going to match up there. But it's not quite. And we'll see in a second what we will need.

So this was, we answered the continuity question. OK. So now let's answer the differentiability question. OK. Notice, again, continuous. We have a lot of lines that will work. OK. But to answer the differentiability question, I need the derivative as I come in from the left to be equal

to the derivative as I come in from the right. So actually, I think I did that backwards. This is, as I come in from the right, I need this derivative-- this gives me a certain value. As I come in from the left, I need a derivative coming this direction. I need them to be the same. That's what it would mean for the function to be differentiable.

I can write that in words. The limit as x goes to 1 from the left of f' of x has to equal the limit as x goes to 1 from the right of f' of x .

I'm not sure-- I think you saw this notation already. Limits coming in from the left are designated by a minus sign. Limits coming in from the right are designated by a plus sign. So we need these two limits to agree in order for the function to have a well-defined derivative at that point.

So let's figure out what this is. Because we have the function here. The function to the right of 1 is $x^2 + 1$. We know its derivative. Its derivative is $2x$. So on this side, we have the limit as x goes to 1 from the right of $2x$. And now we can fill that in. What is that? Well, I just evaluate at $x = 1$ and I get 2. OK?

And now that means I need the limit as x goes to 1 from the left of the derivative to also be 2, so let's fill in this side. The function on the left, at the left of 1 is $a \cdot x + b$. So what is its derivative? Its derivative is just a . Right? Because the derivative of b -- b is a constant. That derivative is 0. The derivative of this is a . OK? So we get the limit as x goes to 1 from the left of a . That's what this left-hand expression is.

Well, if I evaluate a at $x = 1$, I can actually just plug it in. There's no x there. This is a constant function, so as x goes to 1 from the left, I just get a . So that tells me that a has to equal 2 here. OK?

So in order to make the function differentiable, I only have one value of a I'm allowed to use, and that's when a equals 2. OK? So I didn't draw this one, yet. I didn't draw it on purpose because I wanted to save that for last.

So when a is 2, what's b have to be? Well, in order to be differentiable it first has to be continuous. So it has to satisfy this, $a = 2$, and it has to satisfy this, $a + b = 2$. So that tells me $a = 2$, and it tells me b has to equal 0. That comes from our work in number one. OK?

What does that represent? That's a line with intercept 0 and slope 2. So I'll draw that one last. Goes through 0, has slope 2, so it goes through this point. And if I draw those, if I connect those up, we see, if we continued we see that the tangent lines there agree exactly. But the function itself is just this part. It's just, the right-hand side is $x^2 + 1$, the left hand side is $y = 2x$.

So we've now figured out how to make this piecewise function both continuous and differentiable.

And we'll stop there.