PROFESSOR: Hi. Welcome back to recitation. In lecture you've been computing derivatives of functions from the limit definition of derivative. So today we're going to do another example of that and do some graphing, as well.

So I've got a problem written here on the board. So we're defining a function f of x to be 1 over the quantity 1 plus x squared. So what I'd like you to do is graph the function of the curve y equals f of x and to compute the derivative f prime of x from the definition.

So why don't you take a couple of minutes to do that yourself, then come back, and we'll work it out together.

All right. Welcome back. So to start off, let's try graphing this function f of x. So one thing you can always do when you start out graphing a function, is to just plot a few points. And that'll give you a very rough sense of where the function is, at least around those points. So, for example, when x is equal to 0 we have f of 0 is 1 over 1. So that's just 1. So we've got this point here, (0, 1). And when x is equal to 1, well, x squared is 1, so the denominator is 2, so the function value is 1/2.

So, all right I'm not going to draw this to scale. I'm going to put x equals 1 here. And the function value is 1/2, so this is the point (1, 1/2).

And, OK. We could do one more. When x is equal to 2, this function-- 2 squared is 4, so that's 5-- so it's 1/5. So I don't know. 1/5 is smaller than 1/2, right? So that's maybe down here-- so this is something like the point (2, 1/5).

All right. But this is a very rough idea we're getting, so we can use some more sophisticated analysis to get a better idea of what this graph is going to look like. So the first thing we could notice, for example, is that this is an even function. Right? If I change the sign of x, if I replace x by minus x, well, x squared and minus x squared are both equal to x squared. So if you replace x by minus x, the function value doesn't change. So this is an even function that has symmetry across the axis here.

So, you know, for example, I could just mirror image these points. So these points also have to be on the graph, the points minus 1, 1/2 and minus 2, 1/5. And any other part of the curve that I draw will be perfectly mirror imaged.

Another thing to observe is that x squared is always greater than or equal to 0. So 1 plus x squared is always positive. So 1 over 1 plus x squared is also always positive. Also, 1 plus x squared, it reaches its minimum when x is equal to 0 and then as x gets large, either in the positive direction or in the negative direction, this gets larger and larger. This, just the 1 plus x squared part. So the denominator is getting larger and larger, while the numerator stays constant. The whole fraction gets smaller and smaller. So as x gets bigger, either bigger positive or bigger negative, the function value will diminish off to 0, and it has its maximum value here at 0. Because that's when 1 plus x squared has its minimum.

So the function sort of has its maximum here at 0, and then it flattens out. And as x gets larger and larger and larger, this goes to infinity, so the whole fraction goes down to 0. But it never reaches it, right? Because we said it's always positive. And similarly on the other side.

OK. So that's the graph of the curve y equals 1 over 1 plus x squared. Roughly speaking. OK.

So now let's talk about computing the derivative. So right now, to compute a derivative, all you have is the limit definition of the derivative. So when I ask you to compute the derivative what you've got to do is write down what that definition says. That's the limit of a difference quotient.

So we have, by definition, that f prime of x is equal to-- well, it's the limit as delta x goes to 0 of some difference quotient. So on the bottom of the difference quotient we just have delta x, and on the top we have f of x plus delta x minus f of x. In our case, we have a nice formula for f of x. So this is equal to the limit as delta x goes to 0 of 1 over the quantity 1 plus x plus delta x quantity squared-- oh, I guess I didn't need that parenthesis there-- minus 1 over 1 plus x squared, and the whole thing is over delta x.

So, what would be really nice, of course, is if this were a limit where we could just plug in the value delta x equals 0 and evaluate it. But the way the definition of a derivative works, that never works, right? You're always left with the numerator. As delta x goes to 0, that top is always going to be f of x minus f of x and it's going to be 0. And the bottom is always going to be delta x going to 0, which is 0.

So you always, when you have a differentiable function, you always have a derivative that's going to be a limit of a 0 over 0 form. So you need to do some sort of manipulation in order to, in order to get into a form you can evaluate it.

What we'd really like is to manipulate this numerator somehow and pull out, say, a factor of

delta x. And then that could cancel with the delta x we have in the denominator. Something, some trick like that. Some algebraic or other manipulation to make this into a form where we can plug in and evaluate.

So all right, so right here there's sort of only one manipulation that's natural to do, which is we can add these two fractions together. So let's do that, and we can rewrite this limit. The limit is delta x goes to 0. All right, I'm going to pull this 1 over delta x out front just to make everything look a little bit nicer. It's 1 over delta x times-- OK. I want to, you know, subtract these two fractions. I want to put them over a common denominator, so the denominator is just going to be the product of the denominator. So that's 1 plus x plus delta x quantity squared, times 1 plus x squared. OK. And so this fraction is 1 plus x squared over that denominator.

And the second one is 1 plus x plus delta x quantity squared over that common denominator. OK. So we still haven't got where we want to be yet because we still have this 1 over delta x hanging out. So OK, so we have to, you know, keep going.

And so here, I guess there's a-- this is sort of a problem that forces us a little bit in one direction. You, know, there's not much we can do with the denominator, but here in the numerator we can expand this out and start combining stuff.

So let's do that. So this is equal to-- all right, well, the limit hangs out-- the limit as delta x goes to 0 of 1 over delta x times-- OK, so 1 plus x squared minus-- all right, so if you expand out x plus delta x quantity squared, using your favorite, either FOIL or the binomial theorem or just whatever you like, however you like to multiply two binomials-- so we get a minus 1 minus x squared minus 2x times delta x minus delta x squared. That's the top.

OK, and we haven't changed the bottom. It's still 1 plus x plus delta x squared times 1 plus x squared.

OK. Well, so what? OK, so now some nice stuff is starting to happen, which is this 1 and this minus 1 are going to cancel, and this x squared and this minus x squared are going to cancel. And then after we cancel those terms we see that in the numerator here, everything is going to have a factor of delta x, right? These four are going to cancel, and we'll just be left with these two terms, both of which are divisible by delta x. So that's where this cancellation we've been looking for is going to come from.

So let's keep going. So we cancel those, they subtract, give us 0. This limit is equal to the limit

delta x goes to 0-- OK. And then we can divide this delta x from the denominator in, and what we're left with upstairs is minus 2x minus delta x, the whole thing over the same denominator, still. 1 plus x plus delta x squared times 1 plus x squared.

All right. Great. So we've done this manipulation. We finally found a delta x that we could cancel with that delta x we started with in the denominator. And now this limit is no longer this 0 over 0 form, right? When delta x goes to 0, the top goes to minus 2x. And the bottom-- well let's see, this delta x just goes to 0, so it's 1 plus x squared times 1 plus x squared. So that's not 0 over 0. We can just plug in to evaluate. So this is, just works out to a minus 2x over-- OK, 1 plus x squared times 1 plus x squared is 1 plus x squared, quantity squared.

And so this is the derivative that we were looking for. This is, just to remind you what that was, that's d over dx of 1 over the quantity 1 plus x squared.

Now, if you wanted, you could check this a little bit by looking at the graph and looking at this function and just making sure that it makes sense. So for example, this function, this derivative has the property that it's 0 when x is 0. And that's the only time it's 0. And if we go back and look at the graph that we drew over here, we see that's also a property that this graph has, right? It has this horizontal tangent line there, and then it diminishes off to the right and it, on the left side it increases, then it has that horizontal tangent line, and then it decreases.

And so if we go back to this function we see, yes indeed, when x is negative, this whole thing is positive. And then at 0 it's 0, and then it's negative thereafter. And similarly, you could note that this function here is an odd function. If you change the sign of x, that changes the sign of this whole expression, and so OK, and so that makes perfect sense back here.

The symmetry of this curve is such that, you know, if we look at a tangent line to the left of 0 and the symmetric tangent line to the right of 0, they're mirror images of each other. So their slopes are exactly negatives of each other. So that's a nice way you can sort of put the two different pieces of this problem together in order to double check your work. So that's that.