FOURIER SERIES

Let us summarize what we know about power series:

I CONVERGENCE Given a power series

$$\sum_{k=0}^{\infty} a_k x^k,$$

there is a number r with $0 \leq r \leq \infty$, such that the series converges absolutely for $|x| < r$ and diverges for $|x| > r$; we call r the radius of convergence.

II UNIQUENESS If

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

for x in some non-trivial interval about 0, then

$$a_k = f^{(k)}(0)/k! \ .$$

III TAYLOR SERIES If $f^{(k)}(0)$ exists for all k, then we can write down the series

$$\sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = f^{(k)}(0)/k! \ .$$

This series is called the Taylor series of f. It may not converge to f, however; it will do so only if the error term $E_n(x)$ goes to 0 as n approaches ∞. In this case, f is said to be analytic.

IV DIFFERENTIATION AND INTEGRATION If $f(x)$ equals a power series in some non-trivial interval about 0, then $f'(x)$ and $\int_0^x f(t)\,dt$ can be computed by differentiating and integrating the series term-by-term. These new series have the same radius of convergence as the original series,

V APPROXIMATION Among all polynomials of degree n, the Taylor polynomial of f is the one that equals f at 0, and whose first n derivatives equal those of f at 0. If f is analytic, it approximates f very well for x near 0 (and less well as x becomes large).

Now we consider series whose terms are not powers of x, but are of the form $\sin nx$ and $\cos mx$, for n and m positive integers. One motivation for considering such functions is that they are <u>periodic</u>, so they would be natural functions to consider if one wished to represent a periodic function by an infinite series. (Such functions are often called <u>wave functions</u>, and are important in the applications.)

So let us consider a general series of the form

$$\tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) .$$

Such a series is called a <u>trigonometric series</u>. (The factor $\tfrac{1}{2}$ is inserted for later convenience.) We will consider the analogues of statements I - V for this new series.

I CONVERGENCE

About this there is little to say. Trigonometric series have no particularly nice convergence properties. For instance, the series $\sum (\cos nx)/n$ converges at $x = \pi$ and fails to converge at $x = 0$. What happens in between is anybody's guess!

II UNIQUENESS

Here there is a theorem. But since we don't know the series converges on a non-trivial interval, we must assume it.

<u>Theorem 1</u>. If the trigonometric series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges uniformly to a function $f(x)$ on the interval $[-\pi, \pi]$, then

$$a_0 = (1/\pi)\int_{-\pi}^{\pi} f(x)\,dx ,$$
$$a_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and}$$
$$b_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\sin nx\,dx .$$

(since $\cos nx \equiv 1$ if $n = 0$, the first of these equations is redundant.)

<u>Proof</u>. Since the series converges uniformly on $[-\pi, \pi]$, it will still converge uniformly if we multiply through by $\cos mx$ or $\sin mx$. Then we

can compute the integrals of $f(x)$ or $f(x)\cos mx$ or $f(x)\sin mx$ by integrating the appropriate series term-by-term. It happens that if we integrate from $-\pi$ to π, all but one of the terms equal 0! This follows from the integration formulas

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \quad \text{if } n \neq m$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \quad \text{if } n \neq m$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \quad \text{always}$$

$$\int_{-\pi}^{\pi} \sin nx \, dx = 0$$

$$\int_{-\pi}^{\pi} \cos nx \, dx = 0.$$

Finally, the fact that the integrals from $-\pi$ to π of $\cos^2 nx$ and $\sin^2 nx$ equal π gives us the factor of $(1/\pi)$ in the above equations. $\square$

III FOURIER SERIES

Suppose f is an integrable function defined on $[-\pi, \pi]$. Then we can write down the trigonometric series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where the coefficients are given by the integral formulas in the statement of Theorem 1. This series is then called the Fourier series of f. Just as was the case with the Taylor series of a function, however, the series may not converge to f.

The remarkable fact about Fourier series is that they converge under very weak assumptions about the function f, in contrast to the situation for Taylor series, where the function must have derivatives of all orders and, in addition, be analytic. We shall state without proof several theorems concerning the convergence of Fourier series. In order to do so, we must make the following definition:

Suppose that f is continuous on an open interval about p, except possibly at the point p itself. If both the limits

$$\lim_{x \to p+} f(x) \quad \text{and} \quad \lim_{x \to p-} f(x)$$

exist (and are finite), we say that f has at most a jump discontinuity at p.

<u>Theorem 2</u>. Suppose $f(x)$ is continuous on the interval $[-\pi, \pi]$, and that f' is continuous on this interval except for finitely many jump discontinuities. Suppose also that $f(-\pi) = f(\pi)$. Then the Fourier series of f converges uniformly to f on the interval $[-\pi, \pi]$.

<u>Remark</u>. Note that if the Fourier series of f is to converge to f on $[-\pi, \pi]$, then it is necessary that $f(\pi) = f(-\pi)$, since all the functions involved have this property. Note further that if the convergence is to be uniform, it is necessary that f be continuous, since the limit of a uniformly convergent series of continuous functions is continuous.

What is remarkable is the fact that you need to assume very little more than these two necessary conditions in order to ensure that the series converges uniformly. The situation is very different from that for Taylor series!

Now of course if f is <u>not</u> continuous, then there is no hope of getting the Fourier series of f to converge uniformly. Even in this case, however, the series will try as hard as it can to converge! That is the substance of the following theorem:

<u>Theorem 3.</u> Suppose that f and f' are continuous on $[-\pi, \pi]$ except for finitely many jump discontinuities. Then the Fourier series of f converges to f at each point of $(-\pi, \pi)$ at which f is continuous; and the convergence is uniform on any closed interval in $(-\pi, \pi)$ on which f is continuous.

At every point p of $(-\pi, \pi)$, the series converges to the number

$$\tfrac{1}{2}\left(\lim_{x\to p+} f(x) + \lim_{x\to p-} f(x)\right).$$

And at π and $-\pi$, it converges to

$$\tfrac{1}{2}\left(\lim_{x\to -\pi+} f(x) + \lim_{x\to \pi-} f(x)\right).$$

We can understand better what happens at π and $-\pi$ if we note the following: Given $f(x)$, let us look at its values on the half-open interval $[-\pi, \pi)$, and extend f to the entire real line by defining

$$g(x + 2n\pi) = f(x)$$

for all x and all n. The function g is called the <u>periodic extension</u> of f.

Now if the Fourier series of f converges to f for some x in $[-\pi,\pi)$, it will automatically converge to the periodic extension g of f at any point of the form $x + 2n\pi$. In some sense, then, it is more natural to deal with functions $g(x)$ that are of period 2π and defined on the entire real line. It now becomes clear why the Fourier series of f may not converge to f at π or $-\pi$ (even if f, which is only defined on $[-\pi,\pi]$, is continuous there). For the periodic extension g of f will not be continuous at π unless the right and left hand limits of g at π equal the value of g at π.

Restated in these terms, Theorem 3 becomes the following:

Theorem 4. Let $g(x)$ be a function of period 2π, defined for all x. Suppose g and g' are continuous on $[-\pi,\pi]$ except for finitely many jump discontinuities. Then the Fourier series of g has the following properties:

(i) It converges to $g(x)$ whenever g is continuous at x.

(ii) It converges uniformly to g on each closed interval on which g is continuous.

(iii) It converges to the average of the right and left hand limits of g at each point where g is discontinuous.

To illustrate these theorems, we compute some examples. Before doing so, let us recall that we call $f(x)$ an even function if $f(x) = f(-x)$ for all x, and we call it an odd function if $f(x) = -f(-x)$ for all x. The integral of an odd function from $-a$ to a is always 0, because cancellation occurs. The following is an immediate consequence:

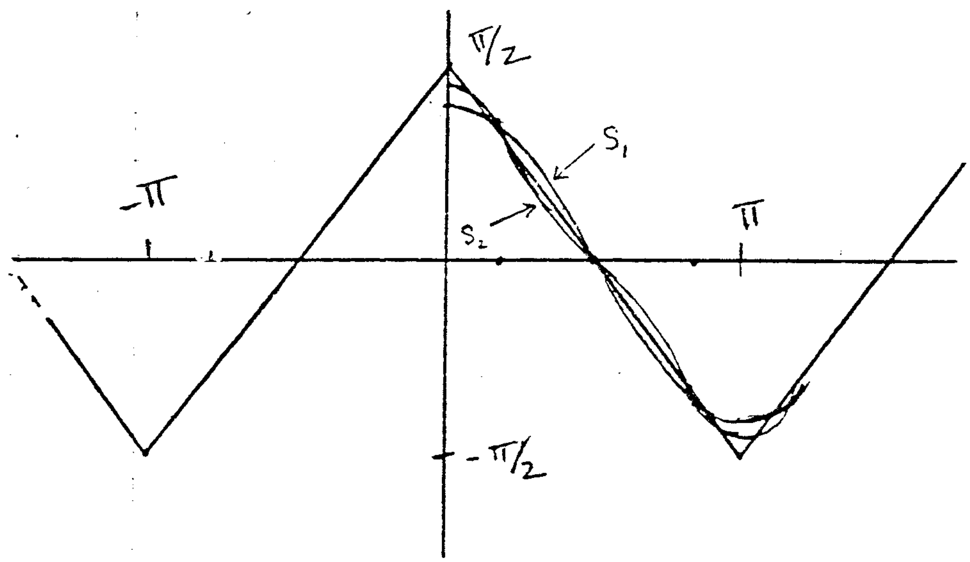
Theorem 5. If $f(x)$ is an even function, then all the terms $b_n \sin nx$ are missing from its Fourier series. If $f(x)$ is odd, then all the terms $\frac{1}{2}a_0$ and $a_n \cos nx$ are missing from its Fourier series.

Proof. If f is an even function, then $f(x)\sin nx$ is odd, while if f is odd, then $f(x)\cos nx$ is odd. $\square$

Example 1. Consider the function

$$f(x) = \tfrac{1}{2}\pi + x \quad \text{for} \quad -\pi \leq x \leq 0,$$
$$f(x) = \tfrac{1}{2}\pi - x \quad \text{for} \quad 0 \leq x \leq \pi.$$

Its graph is pictured; it is called a triangular wave. (We have actually pictured the periodic extension of f.)



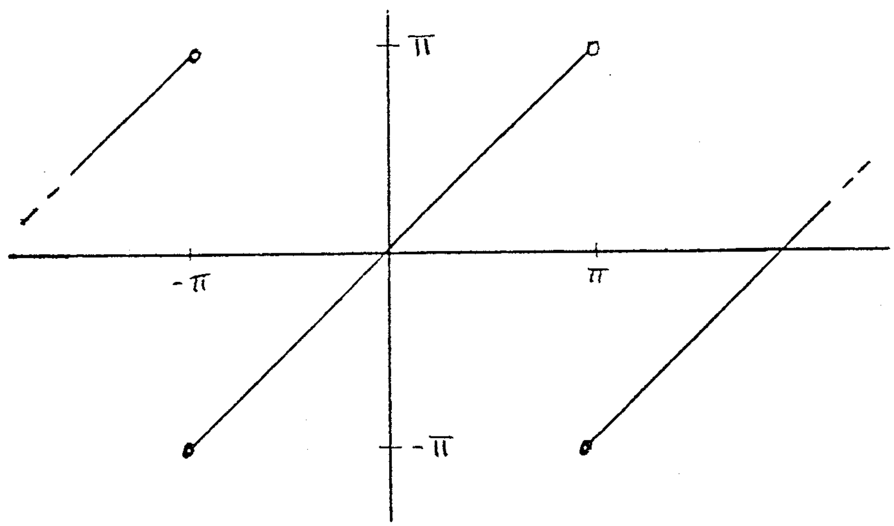
This function is even, so only cosine terms appear in its Fourier series. Direct computation of the coefficients a_n, by integration, gives us the series

$$\frac{4}{\pi}\left[\cos x + (1/9)\cos 3x + (1/25)\cos 5x + \ldots\right].$$

The first two partial sums of this series, s_1 and s_2, are pictured above. Note how closely they approximate the function.

This function satisfies the hypotheses of Theorem 2; this theorem predicts that the series will converge uniformly (since f is continuous and $f(-\pi) = f(\pi)$ and f' has only jump discontinuities). And indeed, it does converge uniformly, by comparison with the series $\sum 1/n^2$.

Example 2. Consider the function $f(x) = x$ for $-\pi \leq x < \pi$. Let g be its periodic extension. The graph of g is pictured below.

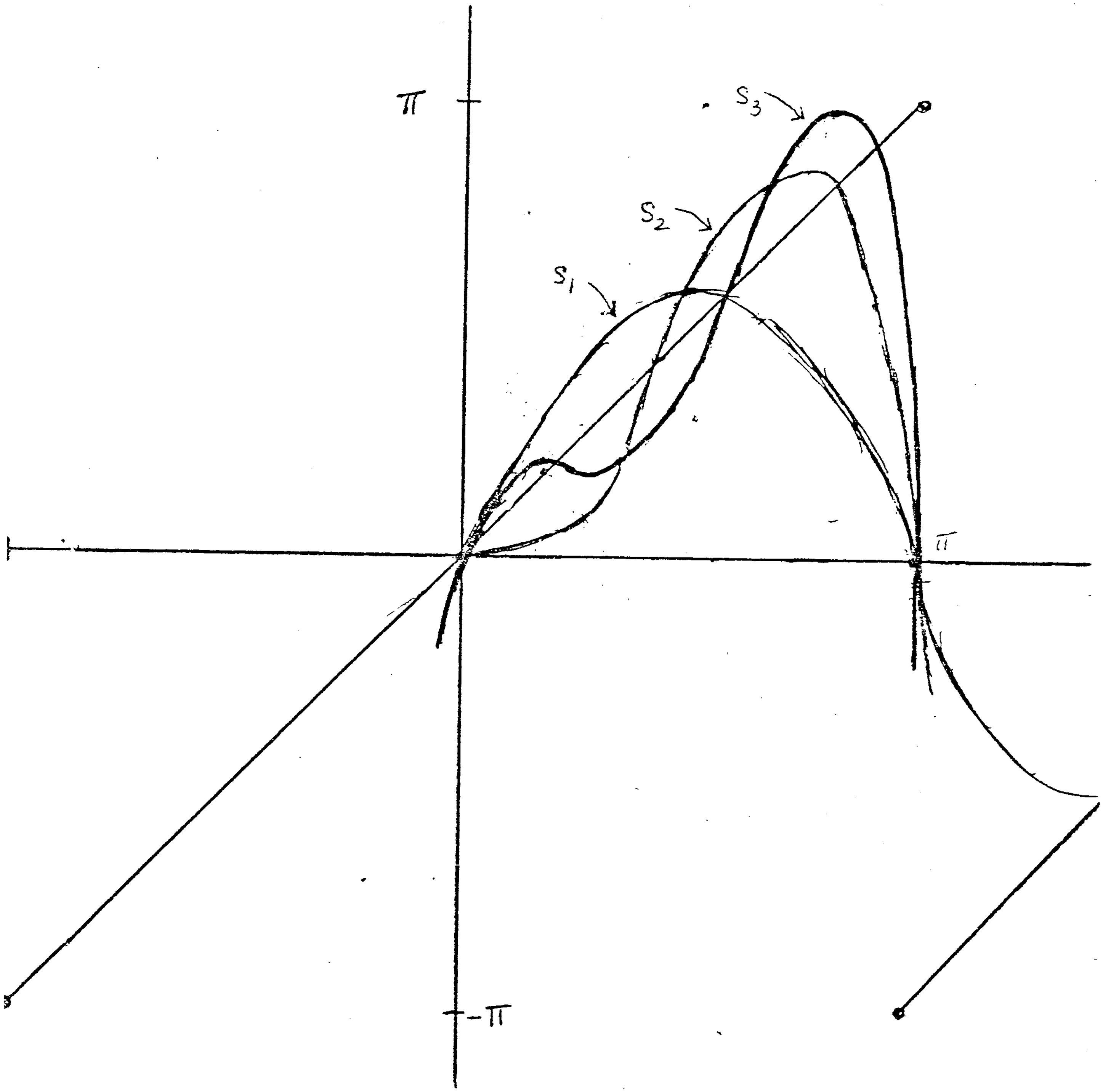


This is a case where the function f is continuous on $[-\pi, \pi)$, but its periodic extension has a jump at π. Thus we do not expect the Fourier series of f to converge to f at π or $-\pi$, but rather to the average of the left and right hand limits, which is 0. And this is exactly what happens.

Since f is an odd function, no cosine terms appear in its Fourier series. Direct computation gives us the series

$$2\left[\sin x - (1/2)\sin 2x + (1/3)\sin 3x - (1/4)\sin 4x + \ldots\right] .$$

The first three partial sums s_1, s_2, and s_3 are pictured in the following figure. Note that the convergence is not nearly as rapid as in the preceding example, and that it gets much worse as one approaches π, where g fails to be continuous.



π
S_3
S_2
S_1
π
-π

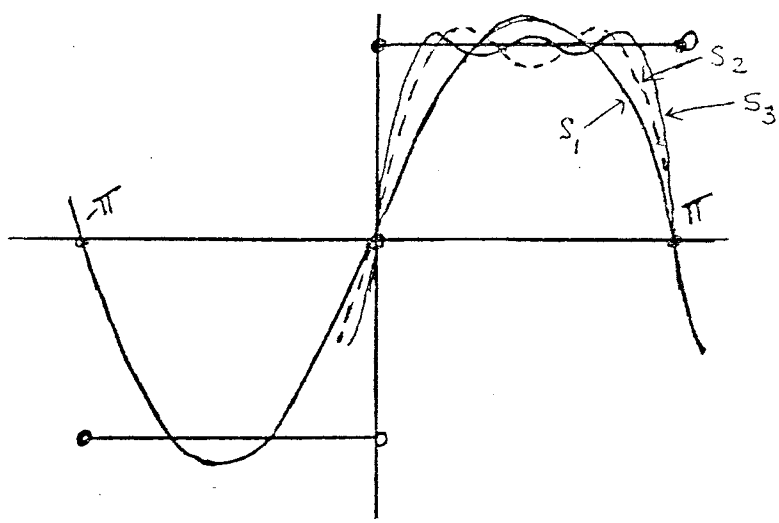
Example 3. Finally, let us consider the following function, which is called the square wave function:

$$f(x) = 1 \text{ if } 0 \leq x < \pi$$
$$f(x) = -1 \text{ if } -\pi \leq x < 0 .$$

This function is also odd; its Fourier series is the series

$$(4/\pi)\left[\sin x + (1/3)\sin 3x + (1/5)\sin 5x + \ldots\right].$$

The first three partial sums are sketched. Note that the convergence becomes worse as one approaches the discontinuities (of the periodic extension of f).



IV DIFFERENTIATION AND INTEGRATION

We know that in general a uniformly convergent series can be integrated term by term. A much stronger result holds for Fourier series; in fact, one does not even need to assume that the series converges!

Theorem 6. Suppose that f is continuous on $[-\pi, \pi]$ except for finitely many jump discontinuities. Although the Fourier series of f need not converge to f, it is still true that if you integrate each term of the series from a to b (where a and b are points of $[-\pi, \pi]$), then the resulting series will converge to the number

$$\int_a^b f(x)\, dx.$$

There is no similar theorem about differentiating a Fourier series.

V APPROXIMATION

Just as was the case for an analytic function and its Taylor series, the Fourier series of a function f will, under quite weak conditions, approximate the function. The difference lies in how one measures the closeness of the approximation. Rather than measuring the actual difference between the values of the function and of the partial sums of the series, we measure the average value, in some sense, of this difference. Specifically, we make the following definition:

Suppose that $f(x)$ is a given function on the interval $[a,b]$. And suppose we seek to approximate f by another function $g(x)$ on this interval. In this case, we call the number

$$E(f,g) = \int_a^b (f(x) - g(x))^2 \, dx$$

the mean square error in this approximation.

One has the following theorem:

Theorem 7. Let $f(x)$ be continuous, except for finitely many jump discontinuities, on $[-\pi,\pi]$. Among all "trigonometric polynomials" of the form

$$h_n(x) = \tfrac{1}{2}a_0 + \sum_{i=1}^{n} (a_i \cos ix + b_i \sin ix) \quad ,$$

the one for which the mean square error $E(f,h_n)$ is a minimum is the one for which the coefficients are the Fourier coefficients of f.

Furthermore, in this situation, the mean square error goes to zero as n approaches ∞.

What this last sentence says is that even though the Fourier series of f may not converge to g in the ordinary sense, it will converge "in the mean."

GENERALIZATIONS

It is remarkable how different the theorems concerning the convergence of power series and the convergence of Fourier series are. It is then natural to ask why the functions $\sin nx$ and $\cos nx$ play such a special role. Perhaps it is because they are periodic. But that is not the case; their periodicity is important only if one wishes to represent periodic functions. If the functions one wishes to represent are not periodic, there are many other systems of functions that will do as well.

The crucial property we needed was that when we multiplied two of the functions $\sin nx$ and $\cos nx$ together and integrated from $-\pi$ to π, we got zero!

For example, it is not at all hard to find a sequence of functions

$P_0(x),\ P_1(x),\ P_2(x),\ldots,P_n(x),\ldots$

such that $P_n(x)$ is a polynomial of degree n, for each n, and such that

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0$$

whenever $n \neq m$. These polynomials are uniquely determined up to a constant factor; it is traditional to multiply each by an appropriate constant so that

$$\int_{-1}^{1} P_n(x)\, P_n(x)\, dx = 1.$$

These polynomials are called the <u>Legendre polynomials</u>. It follows just as in the proof of Theorem 1 that if a series of the form

$$\sum_{n=0}^{\infty} a_n P_n(x)$$

converges uniformly to a function $f(x)$ on the interval $[-1,1]$, then the coefficients a_n are given by the equation

$$a_n = \int_{-1}^{1} f(x)\, P_n(x)\, dx.$$

For any integrable function $f(x)$, the series for which the coefficients are given by this formula is called the <u>Fourier-Legendre series</u> of f. There are theorems about these series that are directly analogous to those about Fourier series mentioned above. Their uses in the applications of mathematics are abundant.