The trigonometric functions.

For the present, we shall assume the following theorem concerning existence of the sine and cosine functions. Later on, when we study power series, we shall prove this theorem.

Theorem 1. *There exist two functions* $\sin x$ *and* $\cos x$, *defined for all real numbers* x, *satisfying the following conditions:*

(i) $\sin 0 = 0$; $\cos 0 = 1$.

(ii) $D \sin x = \cos x$; $D \cos x = -\sin x$.

From the properties listed in this theorem, one can derive all the other familiar properties of the trigonometric functions, as we shall see.

Theorem 2. *Conditions (i) and (ii) specify the functions* $\sin x$ *and* $\cos x$ *uniquely.*

Proof. Step 1. We first note the following fact: If $u(x)$ and $v(x)$ are functions satisfying the equations $u'(x) = v(x)$ and $v'(x) = -u(x)$ for all x, then $u^2 + v^2$ is constant. This result follows from the fact that the derivative of $u^2 + v^2$ is $2uu' + 2vv' = 2uv - 2vu = 0$.

Step 2. We prove the theorem. Suppose $\text{Sin}\, x$ and $\text{Cos}\, x$ are two other functions satisfying these conditions. Let

$$u(x) = \sin x - \text{Sin}\, x \qquad \text{and} \qquad v(x) = \cos x - \text{Cos}\, x .$$

Direct computation shows that $u' = v$ and $v' = -u$. Then $u^2 + v^2 = K$ for some constant K. Substituting $x = 0$, we see that $K = 0$. It follows that $\sin x - \text{Sin}\, x = 0$ for all x and $\cos x - \text{Cos}\, x = 0$ for all x. □

Theorem 3. The functions $\sin x$ and $\cos x$ have the following properties:

(a) $\sin^2 x + \cos^2 x = 1$.

(b) $\sin x$ and $\cos x$ are continuous for all x and have values in the interval $[-1,1]$.

(c) $\int \sin x \, dx = -\cos x + C$; $\int \cos x \, dx = \sin x + C$.

(d) $\sin(x+y) = \sin x \cos y + \cos x \sin y$,

$\cos(x+y) = \cos x \cos y - \sin x \sin y$.

(e) $\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$.

(f) $\sin(-x) = -\sin x$,

$\cos(-x) = \cos x$.

(g) There is at least one positive number x such that $\cos x = 0$.

(h) There is a unique number $a > 0$ such that $\cos a = 0$ and such that $\cos x$ is positive for $0 < x < a$. We commonly denote the number $2a$ by π.

(i) The number π satisfies the inequalities

$$3 \le \pi \le 3.6.$$

(j) $\sin(\pi/2) = 1$; $\sin x$ is strictly increasing on $[0,\pi/2]$. $\cos(\pi/2) = 0$; $\cos x$ is strictly decreasing on $[0,\pi/2]$.

(k) $\sin x$ is strictly decreasing from 1 to 0 on $[\pi/2,\pi]$. $\cos x$ is strictly decreasing from 0 to -1 on $[\pi/2,\pi]$.

(ℓ) $\sin(x+\pi) = -\sin x$; $\cos(x+\pi) = -\cos x$.

(m) $\sin(x+2\pi) = \sin x$; $\cos(x+2\pi) = \cos x$.

Proof. The functions $\sin x$ and $\cos x$ are continuous because they are differentiable. Applying Step 1 of the preceding proof to the functions $\sin x$ and $\cos x$, we see that $\sin^2 x + \cos^2 x$ is K, a constant. Substituting $x = 0$, we have $K = 1$. Parts (a), (b), (c) follow.

To prove (d), define

$$u(x) = \sin(x+a) - \sin x \cos a - \cos x \sin a;$$

$$v(x) = \cos(x+a) - \cos x \cos a + \sin x \sin a.$$

Then $u'(x) = v(x)$ and $v'(x) = -u(x)$, as you can check. It follows that $u^2 + v^2 = K$, a constant. Substituting $x = 0$, one sees that $K = 0$. Then $u(x) = 0$ and $v(x) = 0$ for all x.

(e) These formulas follow at once from (d).

To prove (f) we set

$$u(x) = \cos(-x) - \cos x$$

$$v(x) = \sin(-x) + \sin x.$$

Then $u' = v$ and $v' = -u$, as you can check. It follows that $u^2 + v^2 = K$, a ccnstant. Setting $x = 0$, we see that $K = 0$. Then $u = 0$ and $v = 0$ for all x..

(g) We suppose first that $\cos x > 0$ for all $x > 0$ and derive a contradiction. If $\cos x > 0$ for all $x > 0$, then since $D \sin x = \cos x$, the function $\sin x$ is strictly increasing for all $x \geq 0$. Therefore

$$0 = \sin 0 < \sin x < \sin 2x = 2 \sin x \cos x$$

for all $x > 0$. We can rewrite this in the form

$$0 < (2 \cos x - 1) \sin x$$

for $x > 0$. Since $\sin x$ is positive, then $2 \cos x - 1$ is positive, so $\cos x > 1/2$ for $x > 0$. The comparison theorem for integrals implies that

$$\int_0^b \cos x \, dx \geq \int_0^b 1/2 \, dx, \quad \text{or}$$

$$\sin b \geq 1/2 \, b,$$

for all $b > 0$. This is impossible if $b > 2$.

Therefore $\cos b \leq 0$ for at least one $b > 0$. Since $\cos 0 = 1$, the intermediate-value theorem applied to the interval $[0,b]$ gives us a point x such that $0 < x \leq b$ and $\cos x = 0$.

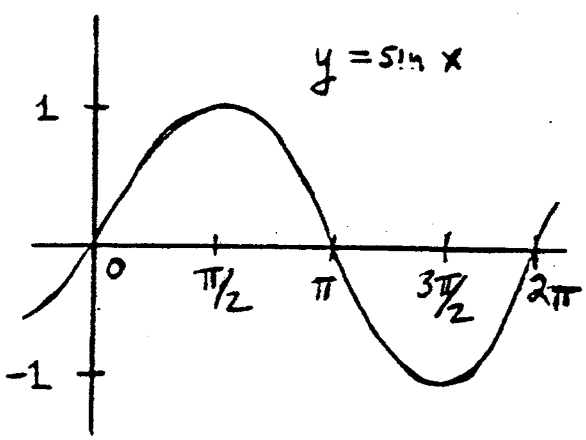
(h) Let a be the inf of the set S consisting of those positive values of x for which $\cos x = 0$. We show that $\cos a = 0$. If $\cos a \neq 0$, then by continuity there is an open interval I about a on which $\cos x \neq 0$. This fact implies that the right hand end point of I is a lower bound for S, a contradiction. Therefore $\cos a = 0$. By choice of a, we know that $\cos x$ is nonzero on the interval $[0,a)$. Because $\cos 0$ is positive, the intermediate-value theorem implies that $\cos x$ must be positive for $0 \leq x < a$.

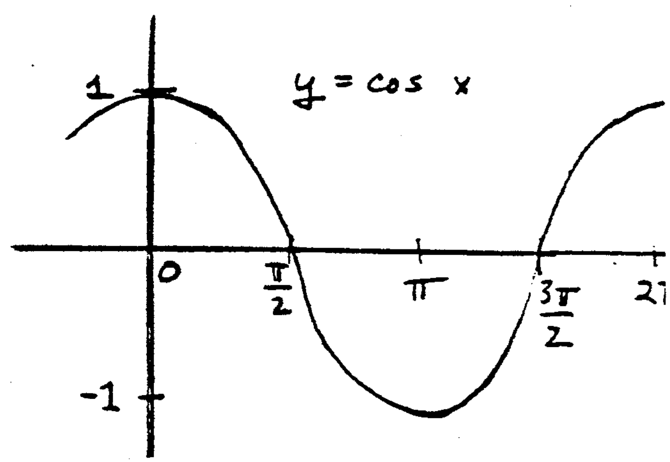
We leave (i) as an exercise.

(j) Because $\cos(\pi/2) = 0$, we must have $\sin(\pi/2) = \pm 1$. Because $\cos x > 0$ on $(0,\pi/2)$, the function $\sin x$ is strictly increasing on $[0,\pi/2]$; therefore $\sin(\pi/2) = +1$. We know $\cos(\pi/2) = 0$; because $D \cos x = -\sin x$ and $\sin x$ is positive on $(0,\pi/2)$, $\cos x$ is strictly decreasing on $[0,\pi/2]$.

Condition (k) follows by computing $\sin(x+\pi/2)$ and $\cos(x+\pi/2)$; conditions (ℓ) and (m) follow similarly. □

<u>Remark</u>. Conditions (j) - (m) suffice to show that the graphs of $y = \sin x$ and $y = \cos x$ are the familiar wave-shaped curves, as you can check.





<u>Definition</u>. We define $\tan x = (\sin x)/(\cos x)$ and $\sec x = 1/\cos x$. This definition makes sense whenever $\cos x \neq 0$; i.e., whenever $x \neq k\pi + \pi/2$, where k is an integer.

<u>Theorem</u> 4. (a) $D \tan x = \sec^2 x$; <u>therefore</u> $\tan x$ <u>is strictly increasing on any interval on which it is defined</u>.

(b) $\tan 0 = 0$; $\tan(-x) = -\tan x$; $\tan x$ <u>is unbounded above and below on the interval</u> $(-\pi/2,\pi/2)$.

(c) $\tan(x+\pi) = \tan x$.

(d) $\tan(x+y) = (\tan x + \tan y)/(1 - \tan x \tan y)$ _if_ $\tan x$ _and_ $\tan y$ _and_ $\tan(x+y)$ _are defined_.

(e) $D \sec x = \sec x \tan x$.

(f) $1 + \tan^2 x = \sec^2 x$.

Proof. The proof is left as an exercise.□

Exercises.

1. Prove the half-angle formulas

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x), \qquad \sin^2 x = \tfrac{1}{2}(1 - \cos 2x).$$

2. Show that $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$.

3. (a) Show that

$$\cos 3x = 4\cos^3 x - 3\cos x.$$

Compute $\cos \pi/6$ and $\sin \pi/6$.

(b) Compute $\cos \pi/3$ and $\sin \pi/3$.

4. Prove Theorem 4.

5. (a) Show that $\sqrt{3}/2 \leq \cos x \leq 1$ for $0 \leq x \leq \pi/6$.

(b) Apply the comparison theorem for integrals to the inequalities of (a) to conclude that $3 \leq \pi \leq 3.6$.

6. Use the half-angle formulas and the substitution rule to compute

$$\int_0^{\pi/3} \sin^2 x \, dx.$$

Remark on motivation

The general pattern of the development of calculus, after the basic theorems have been established, is to introduce and study various functions that arise in the applications.

The most elementary such functions are of course those involving only algebraic operations: the integral powers of x, the polynomial functions, and the rational functions. The next functions one studies arise naturally from certain real-life situations, which are important enough to study in detail. Both the trigonometric functions and the exponential function arise in this way, as we now describe.

In a first course in trigonometry, the sine and cosine functions are usually introduced as functions of an angle, and their study is motivated by their usefulness in "solving triangles", a problem of importance to navigators and surveyors. This approach is of course misleading, for one would never include them in a beginning course in calculus if that were their primary use.

In fact, their importance comes instead from a physical situation called "simple harmonic motion" or "one–dimensional vibration". It arises frequently, and is characterized by the equation

$$f''(x) = -k^2 f(x).$$

This equation is called a differential equation because it is an equation involving an unknown function f(x) and one or more of its derivatives. "Solving" such an equation means finding a function satisfying the equation. This particular equation arises, for example, in describing the motion of a particle acted on by a force that is proportional to the displacement of the particle from its "rest" position. The sine and cosine functions arise in solving this equation; one checks readily that the function sin kx and cos kx do satisfy this equation. But more generally, it is a fact that every solution of this equation can be expressed in terms of these two functions. This we now prove:

Theorem 6. Suppose f(x) is defined for all x, and satisfies the equation

$$f''(x) = -k^2 f(x),$$

where $k \neq 0$. Let $a = f(0)$ and $b = f'(0)$. Then

$$f(x) = a\cos(kx) + b\sin(kx)$$

for all x.

Proof. We show that given k and a and b, there is at most one function f(x) satisfying the conditions:

$$f''(x) = -k^2 f(x),$$
$$f(0) = a,$$
$$f'(0) = b.$$

The theorem follows, since the function $f(x) = a\cos(kx) + b\sin(kx)$ does satisfy these conditions, by direct computation.

So suppose f and g are two functions satisfying the given conditions. Set

$$u(x) = f(x/k) - g(x/k),$$
$$v(x) = \frac{1}{k}[f'(x/k) - g'(x/k)].$$

Then

$$u'(x) = v(x), \text{ and}$$
$$v'(x) = \frac{1}{k^2}[f''(x/k) - g''(x/k)] = -u(x).$$

It follows from the proof of Theorem 2 that $u^2 + v^2$ is constant. Setting $x = 0$, we see that this constant is zero. Thus u and v are identically zero. □

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