PROF. JERISON: So, we're ready to begin Lecture 10, and what I'm going to begin with is by finishing up some things from last time. We'll talk about approximations, and I want to fill in a number of comments and get you a little bit more oriented in the point of view that I'm trying to express about approximations. So, first of all, I want to remind you of the actual applied example that I wrote down last time. So that was this business here. There was something from special relativity. And the approximation that we used was the linear approximation, with a -1/2 power that comes out to be T(1 + 1/2 v^2/c^2).

I want to reiterate why this is a useful way of thinking of things. And why this is that this comes up in real life. Why this is maybe more important than everything that I've taught you about technically so far. So, first of all, what this is telling us is the change in T divided by T, if you do the arithmetic here and subtract T, that's using the change in T is T' - T here. If you work that out, this is approximately the same as 1/2 v^2/c^2. So what is this saying? This is saying that if you have this satellite, which is going at speed v, and little c is the speed of light, then the change in the watch down here on earth, relative to the time on the satellite, is going to be proportional to this ratio here. So, physically, this makes sense. This is time divided by time. And this is velocity squared divided by velocity squared. So, in each case, the units divide out. So this is a dimensionless quantity. And this is a dimensionless quantity.

And the only point here that we're trying to make is just this notion of proportionality. So I want to write this down. Just-- in summary. So the error fraction, if you like, which is sort of the number of significant digits that we have in our measurement, is proportional, in this case, to this quantity. It happens to be proportional to this quantity here. And the factor is, happens to be, 1/2. So these proportionality factors are what we're looking for. Their rates of change. Their rates of change of something with respect to something else.

Now, on your homework, you have something rather similar to this. So in Problem, on Part 2B, Part II, Problem 1, there's the speed of a pitch, right? And the speed of the pitch is changing depending on how high the mound is. And the point here is that that's approximately

proportional to the change in the height of the mound. In that problem, we had this delta h, that was the x variable in that problem. And what you're trying to figure out is what the constant of proportionality is. That's what you're aiming for in this problem. So there's a linear relationship, approximately, to all intents and purposes this is an equality. Because the lower order terms are unimportant for the problem. Just as over here, this function is a little bit complicated. This function is a little more simple. For the purposes of this problem, they are the same. Because the errors are negligible for the particular problem that we're working on. So we might as well work with the simpler relationship. And similarly, over here, so you could do this with, in this case with square roots, it's not so hard here with reciprocals of square roots. It's also not terribly hard to do it numerically. And the reason why we're not doing it numerically is that, as I say, this is something that happens all across engineering. People are looking for these linear relationships between the change in some input and the change in the output. And if you don't make these simplifications, then when you get, say, a dozen of them together, you can't figure out what's going on.

In this case the design of the satellite, it's very important. The speed actually isn't just one speed. Because it's the relative speed of you to the satellite. And you might be-- it depends on your angle of sight with the satellite what the speed is. So it varies quite a bit. So you really need this rule of thumb. Then there are all kinds of other considerations in this question. Like, for example, there's the fact that we're sitting on Earth and so we're rotating around on what's called a non-inertial frame. So there's the question of that acceleration. There's the question that the gravity that I experience here on Earth is not the same as up at the satellite. And that also creates a difference in time, as a result of general relativity. So all of these considerations come down to formulas which are this complicated or maybe a tiny bit more. Not really that much. And then people simplify them enormously to these very simple-minded rules. And they don't keep track of what's going on. So in order to design the system, you must make these simplifications, otherwise you can't even think about what's going on. This comes up in everything. In weather forecasting, economic forecasting. Figuring out whether there's going to be an asteroid that's going to hit the Earth. Every single one of these things involves dozens of these considerations. OK, there was a question that I saw, here. Yes.

STUDENT: [INAUDIBLE]

PROF. JERISON: Yeah. Basically, any problem where you have a derivative, the rate of change also depends upon what the base point is. That's the question. You're saying, doesn't this delta v also

depend, I had a base point in that problem. I happened to decide that pitchers pitch on average about 90 miles an hour. Whereas, in fact, some pitchers pitch at 100 miles an hour, some pitch at 80 miles an hour, and of course they vary the speed of the pitch. And so, this varies a little bit. In fact, that's sort of a second order effect. It does change the constant of proportionality. It's a rate of change at a different base point. Which we're considering fixed. In fact, that's sort of a second order effect.

When you actually do the computations, what you discover is that it doesn't make that much difference. To the a. And that's something that you get from experience. That it turns out, which things matter and which things don't. And yet again, that's exactly the same sort of consideration but at the next order of what I'm talking about here is. You have to have enough experience with numbers to know that if you take, if you vary something a little bit it's not going to change the answer that you're looking for very much. And that's exactly the point that I'm making. So I can't make them all at once, all such points.

So that's my pitch for understanding things from this point of view. Now, we're going to go on, now, to quadratic approximations, which are a little more complicated. So, we talked a little bit about this last time but I didn't finish. So I want to finish this up. And the first thing that I should say is that you use these when the linear approximation is not enough. OK, so, that's something that you really need to get a little experience with. In economics, I told you they use logarithms. So sometimes they use log-linear functions. Sometimes they use log-quadratic functions when the log-linear ones don't work. So most modeling in economics is with log-quadratic functions. And if you've made it any more complicated than that, it's useless. And it's a mess. And people don't do it. So they stick with the quadratic ones, typically.

So the basic formula here, and I'm going to take the base point to be 0, is that $f(x)$ is approximately $f(0) + f'(0) x$. That's the linear part. Plus this extra term. Which is $f''(0)/2 \, x^2$. And this is supposed to work for x near 0. So I've chosen the base point as simply as possible. So here's more or less where we left off last time. And one thing that I said I was going to explain, which I will now, is why it's $1/2 \, f''(0)$. So we need to know that. So let's work that out here first of all.

So I'm just going to do it by example. So if you like, the answer is just, well, what happens when you have a parabola? A parabola's a quadratic. It had better-- its quadratic approximation had better be itself. It's got to be the best one. So it's got to be itself. So this formula, if it's going to work, has to work on the nose, for quadratic functions. So, let's take a

look. If I differentiate, I get b + 2cx. If I differentiate a second time, I get 2c. And now let's plug it in. Well, we can recover, what is it that we want to recover? We want to recover these numbers a, b and c using the derivatives evaluated at 0. So let's see. It's pretty easy, actually. $f(0) = a$. That's on the nose. If you plug in x = 0 here, these terms drop out and you get a. And now, f'(0), whoops that was wrong.

So I wrote f' but what I meant was f. So f(0) is a. Let's back up. f(0) is a, so if I plug in x = 0 I get a. Now, f'(0), that's this next formula here, f'(0), I plug in 0 here, and I get b. That's also good. And that's exactly what the linear approximation is supposed to be. But now you notice, f'' is 2c. So to recover c, I better take half of it. And that's it. That's the reason. There's no chance that any other formula could work. And this one does.

So that's the explanation for the formula. And now I remind you that I had a collection of basic formulas written on the board. And I want to just make sure we know all of them again. So, first of all, there was sine x is approximately x. Cosine x is approximately 1 - 1/2 x^2. And e^x is approximately 1 + x + 1/2 x^2. So those were three that I mentioned last time. And, again, this is all for x near 0. All for x near 0 only. These are wildly wrong far away, but near 0 they're nice, good, quadratic approximations.

Now, the other two approximations that I want to mention are the logarithm, and we use the base point shifted. So we can put it at x near 0. And this one - sorry, this is an approximately equals sign there - turns out to be x - 1/2 x^2. And the last one is (1+x)^r, which turns out to be 1 + rx + r(r-1)/2 x^2. Now, eventually, your mind will converge on all of these and you'll find them relatively easy to memorize. But it'll take some getting used to. And I'm not claiming that you should recognize them and understand them all now. But I'm going to put a giant box around this.

STUDENT: [INAUDIBLE]

PROF. JERISON: Yes. So the question was, you get all of these if you use that equation there. That's exactly what I'm going to do. So I already did it actually for these three, last time. But I didn't do it yet for these two. But I will do it in about two minutes. Well, maybe five minutes. But first I want to explain just a few things about these. They all follow from the basic formula. In fact, that one deserves a pink box too, doesn't it. That one's pretty important. Alright. Yeah. Maybe even some little sparkles. Alright. OK. So that's pretty important. Almost as important as the more basic one without this term here.

So now, let me just tell you a little bit more about the significance. Again, this is just to reinforce something that we've already done. But it's closely related to what you're doing on your problem set. So it's worth your while to recall this. So, there's this expression that we were dealing with. And we talked about it in lecture. And we showed that this tends to e as k goes to infinity. So that's what we showed in lecture. And the way that we did that was, we took the logarithm and we wrote it as k times, sorry, the logarithm of 1 + 1/k. And then we evaluated the limit of this.

And I want to do this limit again, using linear approximation. To show you how easy it is if you just remember the linear approximation. And then we'll explain where the quadratic approximation comes in. So I claim that this is approximately equal to k times 1/k. Now, why is that? Well, that's just this linear approximation. So what did I use here? I used log of 1+x is approximately x, for x = 1/k. That's what I used in this approximation here. And that's the linear approximation to the natural logarithm.

And this number is relatively easy to evaluate. I know how to do it. It's equal to 1. That's the same, well, so where does this work? This works where this thing is near 0. Which is when k is going to infinity. This thing is working only when k is going to infinity. So what it's really saying, this approximation formula, it's really saying that as we go to infinity, in k, this thing is going to 1. As k goes to infinity. So that's what it's saying. That's the substance there. And that's how we want to use it, in many instances. Just to evaluate limits. We also want to realize that it's nearby when k is pretty large, like 100 or something like that.

Now, so that's the idea of the linear approximation. Now, if you want to get the rate of convergence here, so the rate of what's called convergence. So convergence means how fast this is going towards that. What I have to do is take the difference. I have to take ln a_k, and I have to subtract 1 from it. And I know that this is going to 0, and the question is how big is this. We want it to be very small. And the answer we're going to get, so the answer just uses the quadratic approximation. So if I just have a little bit more detail, then this expression here, in other words, I have the next higher-order term. This is like 1/k, this is like 1 / k^2. Then I can understand how big the difference is between the expression that I've got and its limit. And so that's what's on your homework. This is on your problem set.

OK, so that is more or less an explanation for one of the things that quadratic approximations are good for. And I'm going to give you one more illustration. One more illustration. And then we'll actually check these formulas. Yeah, another question.

STUDENT: [INAUDIBLE]

PROF. JERISON: That's a very good question here. When they, which in this case means maybe, me, when I give you a question, does one specify whether you want to use a linear or a quadratic approximation. The answer is, in real life when you're faced with a problem like this, where some satellite is orbiting and you want to know the effects of gravity or something like that, nobody is going to tell you anything. They're not even going to tell you whether a linear approximation is relevant, or a quadratic or anything. So you're on your own. When I give you a question, at least for right now, I'm always going to tell you. But as time goes on I'd like you to get used to when it's enough to get away with a linear approximation. And you should only use a quadratic approximation if somebody forces you to. You should always start trying with a linear one. Because the quadratic ones are much more complicated as you'll see in this next example.

OK, so the example that I want to use is, you're going to be stuck with it because I'm asking for the quadratic. So we're going to find the quadratic approximation near-- for x near 0. To what? Well, this is the same function that we used in the last lecture. I think this was it. $e^{(-3x)}$ $(1+x)^{(-1/2)}$. OK. So, unfortunately, I stuck it in the wrong place to be able to fit this very long formula here. So I'm going to switch it. I'm just going to write it here. And we're going to just do the approximation. So we're going to say quadratic, in parentheses. And we'll say x near 0. So that's what I want. So now, here's what I have to do. Well, I have to write in the quadratic approximation for $e^{(-3x)}$, and I'm going to use this formula right here. And so that's $1 + (-3x) + (-3x)^2 / 2$. And the other factor, I'm going to have to use this formula down here. Because r is $- 1/2$. And so that's $1 - 1/2\ x + 1/2\ (-1/2)(-3/2)x^2$. So this is the r term, and this is the r - 1 term.

And now I'm going to do something which is the only good thing about quadratic approximations. They're messy, they're long, there's nothing particularly good about them. But there is one good thing about them. Which is that you always get to ignore the higher order terms. So even though this looks like a very ugly multiplication, I can do it in my head. Just watching it. Because I get a 1 * 1, I'm forced with that term here. And then I get the cross terms which are linear, which is $-3x - 1/2\ x$. We already did that when we calculated the linear approximation, so that's this times the 1 and this times that 1. And now I have three cross terms which are quadratic. So one of them is these two linear terms are multiplying. So that's plus $3/2\ x^2$. That's -3 times -1/2. And then there's this term, multiplying the 1, that's plus 9/2

x^2. And then there's one last term, which is this monster here. Multiplying 1, and that is -3/8. So the great thing is, we drop x^3, x^4, etc., terms. Yeah?

STUDENT: [INAUDIBLE]

PROF. JERISON: OK, well so copy it down. And you work it out as I'm doing it now. So what I did is, I multiplied 1 by 1. I'm using the distributive law here. That was this one. I multiplied this 3x by this one, that was that term. I multiplied this by this, that's that term. And then I multiplied this by this. In other words, two x terms that gave me an x^2 and a (-3)(-1/2). And I'm going to stop at that point. Because the point is it's just all the rest of the terms that come up. Now, the reason, the only reason why it's easy, is that I only have to go up to x squared terms. I don't have to do the higher ones. Another question, way back here. Yeah, right there.

STUDENT: [INAUDIBLE]

PROF. JERISON: OK. So somebody can check my arithmetic, too. Good.

STUDENT: [INAUDIBLE]

PROF. JERISON: Why do I get to drop all the higher-order terms. So, that's because the situation where I'm going to apply this is the situation in which x is, say, 1/100. So here's about 1/100. Here's something which is on the order of 1/100. This is on the order of $1/100^2$. $1/100^2$, all of these terms. Now, these cubic and quartic terms are of the order of $1/100^3$. That's $10^{(-6)}$. 6. And the point is that I'm not claiming that I have an exact answer. And I'm going to drop things of that order of magnitude. So I'm saving everything up to 4 decimal places. I'm throwing away things which are 6 decimal places out. Does that answer your question?

STUDENT: [INAUDIBLE]

PROF. JERISON: So. That's the situation, and now you can combine the terms. I mean, it's not very impressive here. This is equal to 1 - 7/2 x, maybe, plus 51/8 x^2. If I've made that-- if those minus signs hadn't canceled, I would have gotten the wrong answer here. Anyway. So, this is a 2 here, sorry. 7/2. This is the linear approximation we got last time and here's the extra information that we got from this calculation. Which is this 51/8 term. Right, you have to accept that there's a certain degree of complexity to this problem and the answer is sufficiently complicated so it can't be less arithmetic because we get this peculiar 51/8 there, right. So one of the things to realize is that these kinds of problems, because they involve many, many terms are always

going to involve a little bit of complicated arithmetic.

Last little bit, I did promise you that I was going to derive these two relations, as I said. Did the ones in the left column. So let's carry that out. And as someone just pointed out, it all comes from this formula here. So let's just check it. So we'll start with the log function. This is the function, f, and then f' is $1/(1+x)$. And f'', so this is f', this is f'', is $- -1 / (1+x)^2$. And now I have to plug in $x = 0$. So at $x = 0$ this is ln 1, which is 0. So this is at $x = 0$. I'm getting 0 here, I plug in 0 and I get 1. And here, I plug in 0 and I get -1. So now I go and I look up at that formula, which is way in that upper corner there. And I see that the coefficient on the constant is 0. The coefficient on x is 1. And then the other coefficient, the very last one, is -1/2. So this is the -1 here. And then in the formula, there's a 2 in the denominator. So it's half of whatever I get for this second derivative, at 0. So this is the approximation formula, which is way up in that corner there.

Similarly, if I do it for $(1+x)^r$, I have to differentiate that, I get $r(1+x)^{(r-1)}$, and then $r(r-1)(x+1)^{(r-2)}$. So here are the derivatives. And so if I evaluate them at $x = 0$, I get 1. That's 1^r is 1. And here I get r. $1^{(r-1)}$ times r. And here, I plug in $x = 0$ and I get $r(r-1)$. So again, the pattern is right above it here. The 1 is there, the r is there. And then instead of $r(r-1)$, I have half that. For the coefficient.

So these are just examples. Obviously if we had a more complicated function, we might carry this out. But as a practical matter, we try to stick with the ones in the pink box and just use algebra to get other formulas. So I want to shift gears now and treat the subject that was supposed to be this lecture. And we're not quite caught up, but we will try to do our best to do as much as we can today. So the next topic is curve sketching. And so let's get started with that.

So now, happily in this subject, there are more pictures and it's a little bit more geometric. And there's relatively little computation. So let's hope we can do this. So I want to-- so here we go, we'll start with curve sketching. And the goal here--

STUDENT: [INAUDIBLE]

PROF. JERISON: So that's like 'liner', the last time. That's kind of sketchy spelling, isn't it? Yeah, there are certain kinds of things which I can't spell. But, all right. Sketching. Alright. So here's our goal. Our goal is to draw the graph of f, using f' and f''. Whether they're positive or negative. So that's it. This is the goal here. However, there is a big warning that I want to give you. And this

is one that unfortunately I now have to make you unlearn, especially those that you that have actually had a little bit of calculus before, I want to make you unlearn some of your instincts that you developed. So this will be harder for those of you who have actually done this before. But for the rest of you, it will be relatively easy. Which is, don't abandon your precalculus skills. And common sense. So there's a great deal of common sense in this. And it actually trumps some of the calculus. The calculus just fills in what you didn't quite know yet.

So I will try to illustrate this. And because we're running a bit late, I won't get to the some of the main punchlines until next lecture. But I want you to do it. So for now, I'm just going to tell you about the general principles. And in the process I'm going to introduce the terminology. Just, the words that we need to use to describe what is that we're doing. And there's also a certain amount of carelessness with that in many of the treatments that you'll see. And a lot of hastiness. So just be a little patient and we will do this. So, the first principle is the following. If f' is positive, then f is increasing. That's a straightforward idea, and it's closely related to this tangent line approximation or the linear approximation that I just did. You can just imagine. Here's the tangent line, here's the function. And if the tangent line is pointing up, then the function is also going up, too. So that's all that's going on here. Similarly, if f' is negative, then f is decreasing. And that's the basic idea.

Now, the second step is also fairly straightforward. It's just a second-order effect of the same type. If you have f'' as positive, then that means that f' is increasing. That's the same principle applied one step up. Right? Because if f'' is positive, that means it's the derivative of f'. So it's the same principle just repeated. And now I just want to draw a picture of this. Here's a picture of it, I claim. And it looks like something's going down. And I did that on purpose. But there is something that's increasing here. Which is, the slope is very steep negative here. And it's less steep negative over here. So we have the slope which is some negative number, say, -4. And here it's -3. So it's increasing. It's getting less negative, and maybe eventually it'll curve up the other way. And this is a picture of what I'm talking about here. That's what it means to say that f' is increasing. The slope is getting larger. And the way to describe a curve like this is that it's concave. So f is concave up.

And similarly, f'' negative is going to be the same thing as f concave-- or implies f concave down. So those are the ways in which derivatives will help us qualitatively to draw graphs. But as I said before, we still have to use a little bit of common sense when we draw the graphs. These are just the additional bits of help that we have from calculus. In drawing pictures. So

I'm going to go through one example to introduce all the notations. And then eventually, so probably at the beginning of next time, I'll give you a systematic strategy that's going to work when what I'm describing now goes wrong, or a little bit wrong. So let's begin with a straightforward example. So, the first example that I'll give you is the function $f(x) = 3x - x^3$. Just, as I said, to be able to introduce all the notations.

Now, if you differentiate it, you get $3 - 3x^2$. And I can factor that. This is $3(1-x)(1+x)$. OK? And so, I can decide whether the derivative is positive or negative. Easily enough. Namely, just staring at this, I can see that when $-1 < x < 1$, in that range there, both these numbers, both these factors, are positive. 1-x is a positive number and 1 1+x is a positive number. So, in this range, f'(x) is positive. So this thing is, so f is increasing. And similarly, in the other ranges, if x is very, very large, this becomes, if it crosses 1, in fact, this becomes, this factor becomes negative and this one stays positive. So when x > 1, we have that f'(x) is negative. And so f is decreasing. And the same thing goes for the other side. When it's less than -1, that also works this way. Because when it's less than -1, this factor is positive. But the other one is negative. So in both of these cases, we get that it's decreasing.

So now, here's the schematic picture of this function. So here's -1, here's 1. It's going to go down, up, down. That's what it's doing. Maybe I'll just leave it alone like this. That's what it looks like. So, this is the kind of information we can get right off the bat. And you notice immediately that it's very important, from the features of the function, the sort of key features of the function that we see here, are these two places. Maybe I'll even mark them in a, like this. And these things are turning points. So what are they? Well, they're just the points where the derivative changes sign. Where it's negative here and it's positive there, so there it must be 0. So we have a definition, and this is the most important definition in this subject, which is that is if f'(x_0) = 0, we call x_0 a critical point. The word 'turning point' is not used just because, in fact, it doesn't have to turn around at those points. But certainly, if it turns around then this will happen. And we also have another notation, which is the number y_0 which is f(x_0) is called a critical value. So these are the key numbers that we're going to have to work out in order to understand what the function looks like.

So what I'm going to do is just plot them. We're going to plot the critical points and the values. Well, we found the critical points relatively easily. I didn't write it down here but it's pretty obvious. If you set f(x) = 0, that implies that (1 - x)(1 + x) = 0, which implies that x is plus or minus 1. So those are known as the critical points. And now, in order to get the critical values

here, I have to plug in f(1), for instance, the function is 3x - x^2, so there's this 3 * 1 - 1^3, which is 2. And f(-1), which is 3(-1) - (-1)^3, which is -2. And so I can plot the function here. So here's the point -1 and here's, up here, is 2. So this is-- whoops, which one is it? Yeah. This is -1, so it's down here. So it's (-1, -2). And then over here, I have the point (1, 2).

Alright, now, what information do I get from - so I've now plotted two, I claim, very interesting points. What information do I get from this? The answer is, I know something very nearby. Because I've already checked that the thing is coming down from the left, and coming back up. And so it must be shaped like this. Over here. The tangent line is 0, it's going to be level there. And similarly over here, it's going to do that. So this is what we know so far, based on what we've computed. Question.

STUDENT: [INAUDIBLE]

PROF. JERISON: The question is, what happens if there's a sharp corner. The answer is, calculus is-- it's not called a critical point. It's a something else. And it's a very important point, too. And we will be discussing those kinds of points. There are much more dramatic instances of that. That's part of what we're going to say. But I just want to save that, all right. We will be discussing. Yeah. Question.

STUDENT: [INAUDIBLE] PROF. JERISON: The question that was asked was, how did I know at the critical point that it's concave down over here and concave up over here. The answer is that I actually did not use the second derivative yet. What I used is another piece of information. I used the information that I derived over here. That f' is positive, where f' is positive and where it's negative. So what I know is that the graph is going down to the left of -1. It's going up to the right, here. It's going up here and it's going down there. I did not use the second derivative. I used the first derivative. OK, but I didn't just use the fact that there was a turning point here. So, actually, I was using the fact that it was a turning point. I wasn't using the fact that it had the second derivative, though. OK. For now. You can also see it by the second derivative as well.

So now, the next thing that I'd like to do, I need to finish off this graph. And I just want to do it a little bit carefully here. In the order that is reasonable. Now, you might happen to notice, and there's nothing wrong with this. So let's even fill in a guess. In order to fill in a guess, though, and have it be even vaguely right, I do have to notice that this thing crosses, this function crosses the origin. The function f(x) = 3x - x^3 happens to also have the property that f(0) = 0.

Again, common sense. You're allowed to use your common sense. You're allowed to notice a value of the function and put it in. So there's nothing wrong with that. If you happen to have such a value.

So, now we can guess what our function is going to look like. It's going to maybe come down like this. Come up like this. And come down like this. That could be what it looks like. But, you know, another possibility is it sort of comes along here and goes out that way. Comes along here and goes out that way, who knows? It happens, by the way, that it's an odd function. Right? Those are all odd powers. So, actually, it's symmetric on the right half and the left half. And crosses at 0. So everything that we do on the right is going to be the same as what happens on the left. That's another piece of common sense. You want to make use of that as much as possible, whenever you're drawing anything. Don't want to throw out information. So this function happens to be odd. Odd, and f(0) = 0. I'm considering those to be kinds of precalculus skills that I want you to use as much as you can.

So now, here's the first feature which is unfortunately ignored in most discussions of functions. And it's strange, because nowadays we have graphing things. And it's really the only part of the exercise that you couldn't do, at least on this relatively simpleminded level, with a graphing calculator. And that is what I would call the ends of the problem. So what happens off the screen, is the question. And that basically is the theoretical part of the problem that you have to address. You can program this. You can draw all the pictures that you want. But what you won't see is what's off the screen. You need to know something to figure out what's off the screen. So, in this case, I'm talking about what's off the screen going to the right, or going to the left. So let's check the ends. So here, let's just take a look. We have the function f(x), which is, sorry, $3x - x^3$. Again this is a precalculus sort of thing. And we're imagining now, let's just do x goes to plus infinity.

So what happens here. When x is gigantic, this term is completely negligible. And it just behaves like $-x^3$, which goes to minus infinity as x goes to plus infinity. And similarly, f(x) goes to plus infinity if x goes to minus infinity. Now let me pull down this picture again, and show you what piece of the information we've got. We now know that it is heading up this way. It doesn't go like this, it goes up like that. And I'm going to put an arrow for it, And it's going down like this. Heading down to minus infinity as x goes out farther to the right. And going out to plus infinity as x goes farther to the left. So now there's hardly anything left of this function to describe. There's really nothing left except maybe decoration. And we kind of like that

decoration, so we will pay attention to it. And to do that, we'll have to check the second derivative.

So if we differentiate a second time, the first derivative was, remember, 3 - 3x^2. So the second derivative is -6x. So now we notice that f''(x) is negative if x is positive. And f''(x) is positive if x is negative. And so in this part it's concave down. And in this part it's concave up. And now I'm going to switch the boards so that you'll, and draw it. And you see that it was begging to be this way. So we'll fill in the rest of it here. Maybe in a nice color here. So this is the whole graph and this is the correct graph. It comes down in one swoop down here, and comes up here. And then it changes to concave down right at the origin.

So this point is of interest, not only because it's the place where it crosses the axis, but it's also what's called an inflection point. Inflection point, that's a point where-- because f'' at that place is equal to 0. So it's a place where the second derivative is 0. We also consider those to be interesting points. Now, so let me just making one closing remark here. Which is that all of this information fits together. And we're going to have much, much harder examples of this where you'll actually have to think about what's going on. But there's a lot of stuff protecting you. And functions will behave themselves and turn around appropriately. Anyway, we'll talk about it next time.