18.01 Single Variable Calculus
Fall 2006

Practice Question For Exam 4

1) $\frac{x-4}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4} = \frac{(A+B)x^2+(B+C)x+4A+C}{(x+1)(x^2+4)}$

$\left.\begin{array}{l} A+B=0 \\ B+C=1 \\ 4A+C=-4 \end{array}\right\} \quad \begin{array}{l} A=-1 \\ B=1 \\ C=0 \end{array}$

$\int \frac{x-4}{(x+1)(x^2+4)}\,dx = \int\left[\frac{-1}{x+1} + \frac{x}{x^2+4}\right]dx = -\ln|x+1| + \int \frac{x}{x^2+4}\,dx$

$\boxed{= -\ln|x+1| + \frac{1}{2}\ln|x^2+4| + C}$

2) $\int_0^2 \frac{dx}{(x^2+4)^2} = \int_0^{\pi/4} \frac{2\sec^2 u}{(4\tan^2 u+4)^2}\,du = \frac{1}{8}\int_0^{\pi/4}\cos^2 u\,du = \frac{1}{8}\int_0^{\pi/4}\frac{1}{2}(1+\cos(2u))\,du$

$x = 2\tan u$, $dx = 2\sec^2 u\,du$, $1+\tan^2 u = \sec^2 u$

$= \frac{1}{16}\left(u + \frac{1}{2}\sin(2u)\right)\Big|_0^{\pi/4} = \boxed{\frac{1}{64}(\pi+2)}$

3) ⓐ $\int_0^1 \underbrace{x^{2n}}_{g'} \underbrace{e^{-x^2}}_{f}\,dx = \frac{1}{2n+1}x^{2n+1}e^{-x^2}\Big|_0^1 - \int_0^1 (-2xe^{-x^2})\frac{1}{2n+1}x^{2n+1}\,dx$

$\boxed{n=k-1}$ $\quad \int_0^1 x^{2k-2}e^{-x^2}\,dx = \frac{1}{2k-1}\cdot\frac{1}{e} + \frac{2}{2k-1}\int_0^1 x^{2k}e^{-x^2}\,dx$

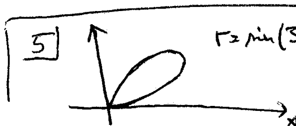
$\boxed{\int_0^1 x^{2k}e^{-x^2}\,dx = \frac{-1}{2e} + \frac{2k-1}{2}\int_0^1 x^{2k-2}e^{-x^2}\,dx}$

ⓑ $\int_0^1 x^2 e^{-x^2}\,dx = \frac{-1}{2e} + \frac{1}{2}\int_0^1 e^{-x^2}\,dx$

4) $V = \int_0^{\pi/2} 2\pi x\cos x\,dx = 2\pi\left[x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx\right]$

$= \boxed{\pi(\pi-2)}$

5)



$A = \int_0^{\pi/3}\frac{1}{2}\sin^2(3\theta)\,d\theta$

$= \frac{1}{4}\int_0^{\pi/3}(1-\cos(6\theta))\,d\theta$

$= \frac{\theta}{4} - \frac{1}{24}\sin(6\theta)\Big|_0^{\pi/3} = \boxed{\pi/12}$

6) a) $L = \int ds = \int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$= \int_0^{\pi/2}\left((-3\cos^2 t\sin t)^2 + (3\sin^2 t\cos t)^2\right)^{1/2}dt$

$= 3\int_0^{\pi/2}\cos t\sin t\,\underbrace{(\cos^2 t+\sin^2 t)}_{1}{}^{1/2}\,dt$

$= \frac{3}{2}\int_0^{\pi/2}\sin(2t)\,dt = -\frac{3}{4}\cos(2t)\Big|_0^{\pi/2}$

$= \boxed{3/2}$

b) $A = \int 2\pi y\,ds = \int_0^{\pi/2} 2\pi\sin^3 t\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{1/2}dt$

$= 2\pi\int_0^{\pi/2}\sin^3 t\cdot 3\cos t\sin t\,dt = 6\pi\int_0^{\pi/2}\sin^4 t\cos t\,dt$

$u = \sin t$
$du = \cos t\,dt$

$= 6\pi\int_0^1 u^4\,du = \boxed{\frac{6\pi}{5}}$

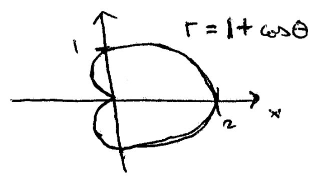
7) $\boxed{L = \int_0^{\pi}\sqrt{1+\cos^2 x}\,dx}$

$\cos^2 x \le 1$, so $\sqrt{1+\cos^2 x} \le \sqrt{2}$

so $\boxed{L = \int_0^{\pi}\sqrt{1+\cos^2 x} \le \pi\sqrt{2}}$

8) $M = \int_0^a 8\cdot 2\pi r\,dr = 2\pi\int_0^a r^3\,dr = \frac{\pi}{2}r^4\Big|_0^a = \boxed{\frac{\pi}{2}a^4}$ grams

9) ⓐ



$r = 3\cos\theta$
$r = 1+\cos\theta$
$3\cos\theta = 1+\cos\theta$
$\cos\theta = 1/2$
$\boxed{(\theta = \frac{\pi}{3}, r = \frac{3}{2}) \quad (\theta = \frac{5\pi}{3}, r = 3/2)}$

9ⓑ i. circle of radius $\frac{3}{2}$ centered at origin:

$r = \frac{3}{2}$, $\frac{3}{2} = 1+\cos\theta$, $\cos\theta = \frac{1}{2}$

$\theta = \pi/3, \frac{5\pi}{3}$ $\quad \boxed{(\theta = \frac{\pi}{3}, r = \frac{3}{2});\ (\theta = \frac{5\pi}{3}, r = 3/2)}$

$r = 3/2$

ii. $(x-\frac{3}{2})^2 + y^2 = (\frac{3}{2})^2$

$x^2 - 3x + y^2 = 0$

$r^2\cos^2\theta - 3r\cos\theta + r^2\sin^2\theta = 0$

$r^2 - 3r\cos\theta = 0$

$r = 3\cos\theta$