

Lecture 8. September 27, 2005

Homework. Problem Set 2 all of Part I and Part II.

Practice Problems. Course Reader: 2A-1, 2A-4, 2A-9, 2A-11, 2A-12.

1. Linear approximations. For a differentiable function $f(x)$, the *linear approximation* or *linearization* of $f(x)$ at $x = a$ is the linear function,

$$f(a) + f'(a)(x - a).$$

In a precise sense, this is the best approximation of $f(x)$ by a linear function near $x = a$. For x close to a , the value of $f(x)$ is close to the value of the linearization. The notation for this is,

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \approx a.$$

Example. The linearization of,

$$f(x) = e^{-3x} \sin(2\pi x) + 5e^{-3x} \cos(2\pi x),$$

near $x = 0$ is,

$$f(x) \approx 5 - (15 - 2\pi)x \text{ for } x \approx 0.$$

In particular, for $x = 0.02$, this gives the approximate answer,

$$f(0.02) \approx 5 - (15 - 2\pi)(0.02) \approx 4.8.$$

The actual value is approximately 4.71.

2. Basic approximations. Some linear approximations occur so often, they should be committed to memory. Each of the following is the linear approximation for $x \approx 0$, together with the terms in the quadratic and higher approximations.

$$\begin{aligned} \frac{1}{1-x} &\approx 1 + x + x^2 + x^3 + \dots, \\ (1+x)^r &\approx 1 + rx + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \dots, \\ \sin(x) &\approx x - x^3/3! + x^5/5! + \dots, \\ \cos(x) &\approx 1 - x^2/2! + x^4/4! + \dots, \\ e^x &\approx 1 + x + x^2/2! + x^3/3! + \dots, \\ \ln(1+x) &\approx x - x^2/2 + x^3/3 - \dots \end{aligned}$$

3. Combining basic approximations. The basic approximations can be combined to get new linear approximations.

(i) The linear approximation of $f(x)$ for $x \approx a$ can be converted to a linear approximation at 0 by setting $g(u) = f(a + u)$. In symbols,

$$f(a) + f'(a)(x - a) = g(0) + g'(0)u.$$

This is equivalent to the formula,

$$\frac{d}{dx}(f(x - a)) = \frac{df}{dx}(x - a).$$

(ii) The linear approximation of $f(cx)$ for $x \approx a$ is obtained from the linear approximation of $f(u)$ for $u \approx ca$ by substituting $u = cx$,

$$f(cx) \approx f(ca) + f'(ca)(cx - ca).$$

This is equivalent to the formula,

$$\frac{d}{dx}(f(cx)) = c \frac{df}{dx}(cx).$$

(iii) The linear approximation of $cf(x)$ for $x \approx a$ is c times the linear approximation of $f(x)$ for $x \approx a$,

$$cf(x) \approx cf(a) + cf'(a)(x - a).$$

This is different than the previous rule. Also, the linear approximation of $f(x) + g(x)$ for $x \approx a$ is the sum of the linear approximations of $f(x)$ and $g(x)$,

$$(f + g)(x) \approx f(a) + g(a) + (f'(a) + g'(a))(x - a).$$

Together, these two rules are equivalent to the formulas,

$$\frac{d}{dx}(cf(x)) = c \frac{df}{dx}(x), \quad \frac{d}{dx}(f(x) + g(x)) = \frac{df}{dx}(x) + \frac{dg}{dx}(x).$$

(iv) The linear approximation of $f(x)g(x)$ for $x \approx a$ is the product of the linear approximations, disregarding all quadratic terms,

$$f(x)g(x) \approx (f(a) + f'(a)(x - a))(g(a) + g'(a)(x - a)),$$

which simplifies to,

$$f(x)g(x) \approx f(a)g(a) + (f'(a)g(a) + f(a)g'(a))(x - a).$$

This is equivalent to Leibniz's rule,

$$\frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}(x)g(x) + f(x)\frac{dg}{dx}(x).$$

(v) The linear approximation of $f(x)/g(x)$ for $x \approx a$ is the quotient of the linear approximations, using the linear approximation $1/(1 - x) \approx 1 + x$,

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)} = (f(a) + f'(a)(x - a)) \frac{1}{g(a)} \frac{1}{1 - (-g'(a)(x - a)/g(a))} \approx$$

$$(f(a) + f'(a)(x - a)) \frac{1}{g(a)} (1 + g'(a)(x - a)/g(a)) = \frac{1}{g(a)^2} (f(a) + f'(a)(x - a))(g(a) + g'(a)(x - a)).$$

This simplifies to,

$$f(x)/g(x) \approx f(a)/g(a) + (1/g(a)^2)(f'(a)g(a) - f(a)g'(a))(x - a).$$

This is equivalent to the quotient rule,

$$\frac{d}{dx}(f(x)/g(x)) = \frac{1}{g(x)} \left(\frac{df}{dx}(x)g(x) - f(x)\frac{dg}{dx}(x) \right).$$

(vi) The linear approximation of $g(f(x))$ for $x \approx a$ is obtained from the linear approximation of $g(u)$ for $u \approx f(a)$ by substituting in for u the linear approximation of $f(x)$ for $x \approx a$ and ignoring quadratic terms,

$$u = f(x) \approx f(a) + f'(a)(x - a),$$

$$g(f(x)) = g(u) \approx g(f(a)) + g'(f(a))(u - f(a)) \approx g(f(a)) + g'(f(a))((f(a) + f'(a)(x - a)) - f(a)).$$

This simplifies to,

$$g(f(x)) \approx g(f(a)) + g'(f(a))f'(a)(x - a).$$

This is equivalent to the chain rule,

$$\frac{d}{dx}(g(f(x))) = \frac{dg}{df}(f(x)) \frac{df}{dx}(x).$$

Together, these 6 rules account for all the general rules we have regarding differentiation. So every rule of differentiation has an equivalent formulation in terms of linear approximations.

Example. Using the rules, the linear approximation for,

$$f(x) = e^{-3x} \sin(2\pi x) + 5e^{-3x} \cos(2\pi x),$$

for $x \approx 0$ is given by,

$$(1 + (-3x))(2\pi x) + 5(1 + (-3x))(1) = 2\pi x + 5 - 15x,$$

which simplifies to,

$$f(x) \approx 5 - (15 - 2\pi)x.$$

4. Quadratic approximations. Sometimes the linear approximation is not good enough. One example is the linear approximation of $\cos(x)$ as 1 for $x \approx 0$. The linear approximation gives no idea whether $\cos(x)$ is greater than 1, less than 1, concave up, concave down, etc. This is remedied by the *quadratic approximation*,

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 \text{ for } x \approx a.$$

Each of the basic approximations has an analogous quadratic approximation. Each of the rules for combining linear approximations has an analogous rule for quadratic approximations.

5. The mean value theorem. This was discussed only very briefly. If a function $f(x)$ is differentiable on the interval having a and b as endpoints, then there is a point c strictly between a and b so that the slope of the tangent line to $y = f(x)$ at $x = c$ equals the slope of the secant line to $y = f(x)$ containing $(a, f(a))$ and $(b, f(b))$,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This is sometimes useful for bounding $f(b) - f(a)$, if a bound on the derivative of $f(x)$ is known.