Lecture 6. September 20, 2005

Homework. Problem Set 2 Part I: (f)–(j); Part II: Problems 1, 3 and 4.

Practice Problems. Course Reader: 1J-1, 1J-2, 1J-3, 1J-4

1. Trigonometric functions. What is *angle*? For a sector of a unit circle (a circle of radius 1), the *angle* of the sector equals both the length of the arc of the sector and 1/2 the area of the sector. Although we have as yet *general definitions* of neither arc length nor area, this can be used to give a rigorous definition of angle. We *can* divide any sector in two equal pieces: simply bisect the chord of the sector. We also know how to add two angles, by laying the sectors in adjacent positions. Denoting the area of a unit circle by the symbol π (which happens to be the familiar π), these 2 operations produce every angle of the form $m\pi/2^n$, with m and n integers. Every angle can be approximated arbitrarily well by such angles. Thus, for every *continuous* function of an angle, every value of the function can be computed.

The basic functions are $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$, $\sec(\theta)$, $\csc(\theta)$ and $\cot(\theta)$. Full descriptions of these are in §9.1 of the textbook by Simmons. The same information is contained in the webpage on Trigonometry produced by MathWorld, part of Wolfram Research.

2. Trigonometric identities. For today, the most important identities are the *angle addition formulas*,

$$\sin(\alpha+\beta) = \boxed{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta),}$$

$$\cos(\alpha+\beta) = \boxed{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).}$$

Other important identities are,

(i) $\cos(-\theta)$ equals $\cos(\theta)$, i.e., $\cos(\theta)$ is an *even function*,

(ii) $\sin(-\theta)$ equals $-\sin(\theta)$, i.e., $\sin(\theta)$ is an *odd function*,

(iii) $\sin(\theta+\pi/2)$ equals $\cos(\theta)$,

(iv) $\cos(\theta+\pi/2)$ equals $-\sin(\theta)$, and

(v) $\sin^2(\theta)+\cos^2(\theta)$ equals 1 for every θ.

3. Some trigonometric limits. In computing trigonometric limits, the following limit is crucial,

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = \boxed{1.}$$

As explained in class, this is essentially the statement that as $\theta \to 0$, the quotient of the arc length by the chord length tends to 1. This was not proved in lecture, nor is it proved in your textbook in §2.1 (despite the author's claim). However, it is geometrically reasonable. And, of course, it can be proved.

This limit implies another limit,

$$\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} = \boxed{0.}$$

To see this, rewrite the term as,

$$\frac{\cos(\theta)-1}{\theta}\frac{\cos(\theta)+1}{\cos(\theta)+1} = \frac{\cos^2(\theta)-1}{\theta\cdot(\cos(\theta)+1)}.$$

By Identity (v), $\cos^2(\theta)-1$ equals $-\sin^2(\theta)$, so the term equals,

$$\frac{-\sin^2(\theta)}{\theta\cdot(\cos(\theta)+1)} = -\frac{\sin(\theta)}{\theta}\frac{1}{\cos(\theta)+1}\sin(\theta).$$

As $\theta \to 0$, this limit tends to,

$$-(1) \times (1/2) \times 0 = 0.$$

By a similar computation,

$$\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta^2} = \frac{-1}{2}.$$

4. Derivatives of $\sin(x)$ **and** $\cos(x)$**.** To compute the derivative of $y = \sin(x)$ at $x = a$, use the angle addition formulas to write,

$$\sin(a + h) = \sin(a)\cos(h) + \cos(a)\sin(h).$$

This gives,

$$\sin(a + h) - \sin(a) = \sin(a)(\cos(h) - 1) + \cos(a)\sin(h).$$

Thus the difference quotient equals,

$$\frac{\sin(a + h) - \sin(a)}{h} = \sin(a)\frac{\cos(h) - 1}{h} + \cos(a)\frac{\sin(h)}{h}.$$

Taking the limit gives,

$$\lim_{h \to 0} \frac{\sin(a + h) - \sin(a)}{h} = \sin(a) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(a) \lim_{h \to 0} \frac{\sin(h)}{h}.$$

Using the limits from above, this gives,

$$\sin'(a) = \sin(a) \times 0 + \cos(a) \times 1 = \cos(a).$$

Thus the derivative of $\sin(x)$ equals,

$$\frac{d\sin(x)}{dx} = \boxed{\cos(x).}$$

An entirely similar computation gives,

$$\frac{\cos(a + h) - \cos(a)}{h} = \cos(a)\frac{\cos(h) - 1}{h} - \sin(a)\frac{\sin(h)}{h},$$

which leads to,

$$\cos'(a) = \cos(a) \lim_{h \to 0} \frac{\cos(h) - 1}{h} - \sin(a) \lim_{h \to 0} \frac{\sin(h)}{h} = \cos(a) \times 0 - \sin(a) \times 1.$$

Thus the derivative of $\cos(x)$ equals,

$$\frac{d\cos(x)}{dx} = \boxed{-\sin(x).}$$

5. Derivatives of other trigonometric functions. Using the quotient rule,

$$\frac{d\tan(x)}{dx} = \frac{1}{\cos^2(x)}(\cos(x) \times \cos(x) - \sin(x)(-\sin(x))) = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}.$$

Therefore, the derivative of $\tan(x)$ equals,

$$\frac{d\tan(x)}{dx} = \boxed{\sec^2(x).}$$

In a similar manner,

$$\frac{d\cot(x)}{dx} = \boxed{-\csc^2(x),}$$

$$\frac{d\sec(x)}{dx} = \boxed{\sec(x)\tan(x),}$$

and

$$\frac{d\csc(x)}{dx} = \boxed{-\csc(x)\cot(x).}$$