Lecture 32. December 9, 2005

Practice Problems. Course Reader: RI.

1. Using power series to solve calculus problems. The reason power series are useful is because they allow us to describe functions that have no direct description. For instance, consider the function,

$$f(x) = \int_0^x e^{-t^2} dt,$$

for $x \geq 0$. By the Fundamental Theorem of Calculus, this function exists and is differentiable with derivative $f'(x) = e^{-x^2}$. Unfortunately, there is no simple expression for $f(x)$ involving only

polynomials, trigonometric functions, exponential functions and logarithms (the proof of this is far beyond the scope of this class). However, it is quite easy to write down a power series expansion for $f(x)$. First of all, the Taylor series for e^{-t^2} about $t = 0$ is obtained by substituting $x = -t^2$ in the Taylor series for e^x about $x = 0$,

$$e^{-t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}/n!.$$

Because this series converges absolutely, the integral of the series is the series of the term-by-term integrals,

$$f(x) = \int_0^x e^{-t^2} dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} dt = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n}{n!} t^{2n} dt.$$

Each of these integrals can be computed quite easily. This gives,

$$f(x) = \boxed{\sum_{n=0}^{\infty} (-1)^n/[(2n+1) \cdot n!] t^{2n+1}.}$$

This is the Taylor series expansion for $f(x)$ about $x = 0$. For instance, using this series, it is easy to estimate,

$$\int_0^1 e^{-t^2} dt \approx 0.747 \pm 10^{-3}.$$

2. Taylor series with remainder term. As demonstrated by the computation just done, in reality only finitely many terms in a Taylor series are used. What can be said in this case? In other words, how quickly does the series converge? How large is the remainder after n terms? To make all this precise, introduce the function $R_{N,a}(x)$ defined to be,

$$R_{N,a}(x) = f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

This is precisely the remainder term so that we have,

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x-a)^n + R_{N,a}(x).$$

The precise version of the questions above is, what bounds exist for $R_{N,a}(x)$?

To understand the answer, consider the simplest case where $N = 0$. Then the remainder term is simply,

$$R_{0,a}(x) = f(x) - f(a).$$

By the Mean Value Theorem, for every x there exists a real number c (depending on x) between a and x such that,

$$R_{0,a}(x) = f'(c)(x-a).$$

Iterating the Mean Value Theorem, for every integer N, for every x, there exists a real number c (depending on both N and x) between a and x such that,

$$R_{N,a}(x) = \boxed{f^{(N+1)}(c)(x-a)^{N+1}/(N+1)!.}$$

In particular, if we can bound the $(N+1)^{\text{st}}$ derivative of $f(x)$ on the interval between a and c, then we can bound $R_{N,a}(x)$.

Example. Bound the remainder in the Taylor series expansion for e^x about $x = a$. The $(N+1)^{\text{st}}$ derivative is simply e^x. Therefore, a bound for $f^{(N+1)}(c)$ for c between a and x is simply,

$$M = e^m = e^{\max(a,x)}.$$

This is independent of N. The bound on the remainder term is then,

$$|R_{N,a}(x)| \leq \boxed{M(x-a)^{N+1}/(N+1)!.}$$

By choosing N suitably large, we can make this remainder term as small as possible. For instance, if we want to compute e^x for x in the interval $(-1, 1)$, then M equals e. To make the remainder term less than 10^{-10}, it suffices to take $N = 12$.

3. Review problems. Each of the following problems was discussed in lecture. Here are the problems and answers, without the discussion.

Problem 1. Let a and b be positive real numbers. There are 2 tangent lines to the ellipse with equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

containing the point (a, b). Find the equations of each of these tangent lines.

The 2 tangent lines are the line tangent to the ellipse at $(x, y) = (0, b)$ and the line tangent to the ellipse at $(x, y) = (a, 0)$. The equations of these lines are,

$$\boxed{y = b,}$$

and,

$$\boxed{x = a.}$$

Problem 2. A grain silo is designed by attaching a cylinder of fixed radius r and height a directly above a right circular cone of base radius r and height b. The silo has no top, and there is no bottom between the bottom of the cylinder and the top of the cone. For a fixed volume V, what choice of b minimizes the surface area of the grain silo?

The choice of b minimizing the surface area is,

$$b = \boxed{2\sqrt{5}r/5.}$$

Problem 3. Compute the volume of the solid obtained by revolving about the x-axis the region in the first quadrant bounded by the curve $y = x^2$ and the curve $x = y^2$.

The volume of this solid is,

$$\text{Volume} = \boxed{3\pi/10.}$$

Problem 4. Using a trigonometric substitution and a trigonometric identity, compute the antiderivative,

$$\int \frac{\sqrt{1 - x^2}}{x^2} dx.$$

The antiderivative equals,

$$\int \frac{\sqrt{1 - x^2}}{x^2} dx = \boxed{-\sqrt{1 - x^2}/x - \sin^{-1}(x) + C.}$$

Problem 5. Using integration by parts, compute the following antiderivative,

$$\int x \sin^{-1}(x) dx.$$

The antiderivative equals,

$$\int x \sin^{-1}(x) dx = \boxed{-(1/4)(1 - x^2)\sin^{-1}(x) + (1/4)x\sqrt{1 - x^2} + C.}$$