PROFESSOR: OK, its always disappointing when, just as you're ready to tie everything up in a nice, neat little package and leave the room to a round of applause, you screw up the last little detail.

Anyway, what's wrong here is, I said for an arbitrary rotation around X3, and I'm comforted only by the fact that nobody else caught this little glitch either, the answer elements are a11, a11, a33, a12, not 0, and -a12, 0, 0, 0, 0. So there's a skew-symmetric form when there's just a single rotation axis.

So this would be asymmetry n and this would be one of a family of two different symmetries, the dihedral groups n22, the two-fold axes would require that these be equal, so, 0, 0, 0, a11, 0, 0, 0, a33. So this is not the form of the tensor for a four-fold axis, it's the form for a crystal with symmetry 422. And same for a three-fold axis. This is the form for 32 and 622.

For just the single rotation axis of any sort other than 2, these two terms are equal in magnitude, but non-0. Now if you look at the equation for the tensor, the term in a12, x1, x2, and a21, x1, x2 are opposite in sign and disappear. So the quadric is, in fact a surface of revolution

So the question is why are there two degrees of freedom rather than three? And I think the answer to that is going to lie in what is required of the eigenvectors when the off diagonal turns are skew-symmetric, and that is something I've not thought about before. All right, so that one minor loose end sticking out of the lid as we try to close the box on this segment of our discussion.

I'd like t move on to a different set of considerations. One of the things that we can subject a crystal to as a generalized force is not merely a vector force, but stimulations that are more complicated. And I would like to in particular now, as a prelude to get into getting into property tensors of higher rank.

I'd like to . first discuss stress, and then define strain, and you've all heard of these questions in other contexts. I'm going to introduce it, though, in terms of the tensor notation that we've used for other sorts of tensors, just so that we have everything in terms of the same language.

So let me introduce stress by supposing I have a volume element in a solid, and it will not surprise you that I'm going to define the coordinate system in the solid by three axes, X1, X2, X3 in a Cartesian. system, and I will look at a surface on the exterior of the solid that cuts axis X1 at A, axis X2 at B, and axis X3 at C.

Then I'll assume that I have a force density, force per unit area applied to the surface. And if this is force per unit area, why don't I call it a pressure? Well, a special case of what we normally think of as a pressure is a hydrostatic pressure, and that is something that's always exactly normal to the surface.

So I don't want to call it a pressure to mislead you into the assumption that this force per unit area, which is a vector, has to be normal to the surface. So I'll call it a force density, which becomes a hydrostatic pressure when that force vector is normal to the surface. And I'll call that force density vector K, and it will have three components, KI.

Now, the solid is in equilibrium, I will assume. And if that external force is not balance on the other side by balancing forced, the volume element in the solid would undergo a linear acceleration, or an angular acceleration at the very least. So I will assume that this force on the outside, KI, which is a force per unit area.

So to make that a force, I will have to multiply that force density by the area of the surface ABC, on which it acts. And I'll say that this has to be balanced for each of the three directions, X1 and X2 and X3 by a force per unit area acting on these internal surfaces. So I'll assume that there is a force acting in the x1 direction, and I'll call those a sigma IJ.

And I'll say that there's a first force acting in the X1 direction on-- now I need to somehow specify the orientation to the surface, because on this internal surface there's also a force acting in this direction. Now on this back surface there's also a force acting in this direction.

So the forces acting on all three internal surfaces in the X1 direction have to balance exactly the X1 direction, the X1 component of the force density. So let me turn this to K1, and I'll say this has to be a force per unit area acting on area. First of all, I'll call the origin O on area AOC. Plus a force acting in the X1 direction on area A-- whoops, what did I do?

I wanted to do this first on area BOC, excuse me. And a force in the X1 direction acting on area AOC. And finally a force acting on area AOB. And these are all acting in the X1 direction, and I now would like to introduce some notation that indicates the direction of the direction orientation of the surface on which that force acts.

And I will use the normal to these internal surfaces to define that. So the force per unit area on area AOB, that is a surface whose normal is X3. And I will call this component sigma 13. And I'll call this one area AOC is normal to X2. That's why I did that little sleight of hand as I began this, and finally area BOC has as its normal X1.

So my use of subscripts here is that this first subscript is the direction in which the force acts. And the second subscript will be the normal to the internal surface. So I've set up algebraically a system of subscripts that works. We'll see that these have some physical significance in just a moment.

So what will I call these? Well, let me get this in a slightly different form. Let me take the left hand side and divide area ABC into all of these other areas. So this then will have sigma 11 times area BOC over area ABC. And not surprisingly, sigma 12 will then have multiplying it a term area AOC divided by area ABC.

And this is all looking very cumbersome, but you'll be amazed at how this cleans up in just a moment. And this will be area AOB over finally area AOC. ABC again. OK, so how can we tidy this up? Well, let me imagine that I have some surface, a planar surface, and its normal points in this direction.

And I'm going to take that area and project it onto another surface whose normal points in this direction. And let's suppose that the angle between these two directions is phi, and the angle between the two surfaces is phi. The area of this original surface is A, and its normal points in this direction.

And I project it onto a surface whose normal is an angle phi away. This new area, A prime is equal to A times the cosine of phi. So if we accept that, then the first line of this equation, which will be eventually a set of three equations, this is the ratio of area BOC over area ABC, and what is that?

That is the angle between the direction of the force density and area ABC. And here is area ABC. That's the normal here. And the first term involves area BOC, and that is the direction of X, the normal to that is X1, and therefore the ratio of these two areas is the same thing as the cosine of the angle between K and X1.

And this second ratio of areas is going to be sigma 12 times the cosine of the angle between the force density vector K and X2. And the third term is going to be sigma 13 times the cosine of the angle between K and X3. What these cosines are are just the cosines of the angle between K and the three reference axes, X1, X2, and X3.

So this is simply the direction cosine L1 of K. And this is the direction cosine L2 of K. And this is the direction cosine L3. So the X1 component of the force density vector is going to be equal to a term sigma 11 times L1, plus the term sigma 12 times L2, the term sigma 13 L3.

And if I would write down similar balances, forces for the X2 component of the force density vector, not surprisingly these things are going to give me coefficients that I'll label sigma 121, sigma 22 times L2, plus sigma 23 times L3.

And in general then the requirement that this body be in equilibrium states that each component of the force density vector should be given by a coefficient sigma IJ times L sub J. These are the components of the force density vector. This is the direction cosines of the direction of K.

This is a unit vector in the direction of K. This is the direction of a vector. These are the components of K. And therefore these coefficients sigma IJ relate to vectors, and they must be a second rank tensor. And these are called, as you all know, it comes as no surprise as the components of stress.

So the reason that these are really tensors, and not just a vector force per unit area essentially comes down to the fact that the behavior of the body, dynamically and mechanically, is going to depend not only on the direction of the force, but on the direction of the surface on which it acts.

Quite clear here, if I shove, is a force density, and that's a vector. But the way in which this desk responds to my pushing on it with a force vector of that magnitude is going to depend whether I do this, which is going to compress it, or whether it takes the same force and direct it this way, which if I don't move the thing is going to result in a shear.

So the behavior of the body will depend on the direction of the force per unit area that I apply, and the direction of the surface on which I apply it. In this case, that would be the surface ABC. Yes, sir?

AUDIENCE: You say that the force density is not parallel to the normal of the surface?

PROFESSOR: That's correct.

AUDIENCE: So when you do the ratio of [INAUDIBLE] isn't it supposed to be [INAUDIBLE] of the normal to the [INAUDIBLE] and X1?

PROFESSOR: Another way of defining it. This is a vector, and what I'm doing is splitting that vector up into three internal forces that act on surfaces whose normal are here. So actually, what I'm going to do is to take the way in which the internal force pushes back. I see, that's what's confusing.

If the force that I'm putting on the body, I didn't mean to indicate any particular direction, but if this component of K points in this direction, this force that balances it has to go in the opposite direction.

AUDIENCE: So you're saying that there were surface ABC is defined as normal to K?

PROFESSOR: ABC is not normal. No. ABC is not normal to K. All I'm saying is that it if it were normal to K, then the balancing force would be just a function of this orientation of the surface. But if I allow this to be a general force per unit area, then the balancing force on the inside depends on the angle between K and the angle between this internal surface.

AUDIENCE: I don't see the relation between K and the ratio between other ends. BOC and ABC [INAUDIBLE].

PROFESSOR: OK. The reason that there are three areas, is that what's troubling you? Because I'm really now splitting the balancing force up into a force per unit area on this surface, and a force per unit area on this surface, and a force per unit area on this surface. And that's so that I can look at these different components of resistance in terms of a Cartesian system.

Clear that the ratio of these two areas does go as the-- I erased it now, that is going to go as the-- This is A prime, and this is the original A, the relation between those areas is given by the cosine of this angle phi. If I [INAUDIBLE] this down in this projection, and this is the area of A.

This is the direction of the normal to the surface ABC. And I'm going to let this be the direction of the normal to one of the internal surfaces, and these internal surfaces are going to have as their normal either X1, X2, or X3.

AUDIENCE: So [INAUDIBLE] between the normal to ABC and X1.

PROFESSOR: And X1, yes. And that's what I'm saying its, and that is going to be the same thing. As the cosine of that angle is just going to be the direction cosine of the normal to ABC relative to X1, X2, X3. OK?

AUDIENCE: So in the end, you're saying that K is parallel to the normal of ABC?

PROFESSOR: No, I'm not. No, I'm not. This is a relation between the areas. And these are the forces acting on those internal surfaces. And I'm saying there are three of them, and those three components that are all acting in the X1 direction will depend on the ratio of these areas. The ratio of this area is the cosine of the angle between X1 and K. OK. This you're OK with?

AUDIENCE: Yeah.

PROFESSOR: OK. And what I'm saying now is that there is acting on this internal surface something that has to balance the component of K, which points in the direction of X1. And the component of K before we divided out the area, the component of K1 times the area gave me the net force.

So this K1 now is a force per unit area, the component of K per unit area that points along the X1 direction. Now as a force per unit area I multiply it by an area that gave me force. There are three forces internally that balance this force, and they all act in the X1 direction.

AUDIENCE: You're just [INAUDIBLE]. Shouldn't it be the cosine of the angle between the normals of [INAUDIBLE]?

PROFESSOR: Yeah. That's what I'm saying here. That for X, for this surface, yeah, that's what I'm saying. One of these surfaces here, if we look at the first term, this is the direction of X1, and the force here will be the force along K. And if I take this force and put it down in X1, this is going to be the force times the cosine of the angle. And the two areas are going to go as the cosine of the angle.

AUDIENCE: [INAUDIBLE] cosine of the normal [INAUDIBLE]?

PROFESSOR: OK. This is indeed the ratio between the internal area and the external area. Since the internal area has a normal that's either X1, X2, or X3 the angle between the normal to these internal surfaces and K is going to be the direction cosine of K.

I suggest since we're getting out of time quickly we either resolve this after the end of class or leave it until next time. That's a bad note on which to end, too. The point I want to leave, still subject to resolution and debate is that the coefficient sigma IJ I would like to claim relate to vectors.

One is the direction of the external surface, and if it relates to vectors, then this set of coefficients qualifies as a second rank tensor. And if you accept that, however grudgingly, everything that we've said about second rank tensors holds for the quantity sigma IJ, which are called the elements of stress.

In particular, we can talk about a stress quadrant. We can talk about the value of a stress in a particular direction. If we have a stress tensor in which show all the terms are non-0, it can be diagonalized.

We can also say that if the crystal has symmetry, for example, if the crystal is cubic, you can say that the form of a second rank tensor for a cubic crystal has to be diagonal. And these diagonal elements represent compressive stresses, so you can say that you can only subject a cubic crystal to compressive stress.

On the other hand, if the crystal is a triclinic crystal, you can say that there's a sigma 11, there's a sigma 12, a sigma 13, a sigma 21, a sigma 22, sigma 23, sigma 31, sigma 32, sigma 33.

So for a triclinic crystal, there appear to be nine elements, but next time we will show that the nature of stress is the body has to be in equilibrium requires that the stress tensor be by definition symmetric if there's to be no net couple to force on it. Going to buy this? You didn't buy the things that I said that perhaps were slightly incorrect. This is massively incorrect. Jason?

AUDIENCE: You can rotate a cubic crystal [INAUDIBLE].

PROFESSOR: No. Because say a tensor of this form acts just like the scalar. Just like a cubic crystal, you don't suddenly get off diagonal properties if you transfer it to a different axis. So unless you tell me what's wrong here, I'm going to leave this on your plate until next week. And I don't want you fretting over the weekend.

This is clearly nonsense, and it is a contradiction that arises only if you carry too much of what we've been doing for the last month with you into a discussion of stress and strain. Stress is something that you impose on a crystal. It has nothing to do with what kind of crystal it is and what's going on in the interior of the crystal.

There may be constraints on the coefficients that relate stress and strains, and there surely are. But I can take a cubic crystal, and I can squeeze it, and I can twist it and share it anyway I like. So this is an important distinction between the tensors that represent physical properties, which we've been discussing up to this point.

And the properties such as conductivity, diffusivity, susceptibility and so on and all their many varieties, these are something which we'll refer to as property tensors. And they are tensors demonstrably, but they are tensors which describe the physical behavior of a crystal.

Nye uses a term that I don't think is self explanatory, so I don't care for it. He calls these field tensors. That sounds like some description of a corn patch that requires a tensor. But no. We probably don't really think of what we mean when we say a field.

When we say an electric field, you think of an e-vector. You think of a vector. But what a field is is just a set of coordinates, XYZ, and that describes an area or a volume. And just like the things that corn grow in are referred to as corn fields, it's an area that has corn on a lattice very often also.

So a field is just a set of coordinates in space to which a value of something is a sign. So if we have an electric field, and we express that electric field as a function of X, Y, and Z, that is a field of vectors.

If we take something that has intrinsically, that intrinsically needs to be described as a tensor, then we have to every point in space XYZ a tensor sigma IJ, whose value and whose components value vary with position within the space. So talking about the field as assigning values of something to every coordinate in the space that could be a tensor.

What we refer to as an electric field really is a field vector. So that's Nye's term, but the fact that it took me three minutes to explain why he uses it explains as well why I don't care for it. So we'll talk about these as property tensors. And property tensors are something that's innate to the material.

Something like stress, strain, and other second rank tensors are external stimuli, just like an electric field. And they have value that is imposed on the crystal, and the stress field is defined as a function of position.

So that's the distinction between a property tensor, which is subject to symmetry constraints, and a tensor that represents a stimulus, an external stimulus applied to the crystal, and it's defined as a function of position. And that is a field tensor.

OK, so that's the difference. Since this is something that's imposed externally, it can have any form of the tensor that you wish to oppose. And there are no symmetry restrictions on it.

The only restriction is, as we'll show next time, that it must be symmetric if the body is to be in equilibrium. All right. It is time to quit, and that I think is a good place to leave things. And we'll say more about stress next time.