3.185 Final Exam

Adam Powell

Solutions

Part A: Closed Book

1. Time Scale and Fully-Dveloped Flow

   (a) These timescales are respectively:

   $$t_{SS,MT} = \frac{L^2}{D},\ t_{SS,HT} = \frac{L^2}{\alpha},\ t_{SS,FF} = \frac{L^2}{\nu}.$$

   (b) The distance traveled at velocity u_{av} over time t_{SS} is just their product:

   $$L_e \sim u_{av} t_{SS} = u_{av} \frac{L^2}{\nu}.$$

   (c) The real entrance length, which is approximately

   $$L_e = \frac{H^2 u_{av}}{100\nu},$$

   is two orders of magnitude shorter than that derived in part 1b. This is for two reasons:

   - The fully-developed flow profile looks similar to the profile when the boundary layers meet each other, so we don't need to wait for full equilibration all the way across the channel, just for the top and bottom to influence the middle. If the boundary layer solution were an erf, we'd need the timescale of erf validity for half of the thickness, which is:

     $$t_{erf} = \frac{(H/2)^2}{16\nu} = \frac{H^2}{64\nu}.$$

   - The Blassius boundary layer solution grows more than 25% faster than the erf solution (5.0 vs. 3.6), because of convective momentum transfer away from the sides and toward the center of the stream.

   All of this adds up to an entrance length 100 times shorter than the estimate in part 1b.

2. Biot and Nusselt Numbers

   (a) For heat transfer, the Biot and Nusselt number are given by:

   $$\text{Bi} = \frac{hL}{k_{sol}} \text{ and } \text{Nu}_L = \frac{hL}{k_{fl}}.$$

   There are two important differences between these two expressions:

- The Biot number uses the solid thermal conductivity, and the Nusselt number the fluid conductivity.
- The length scale L in the Biot number is the thickness of the solid, in the Nusselt number it's the length of the solid plate.

(b) The Biot number is a ratio of resistances to heat transfer by conduction through a solid and convection in a fluid in series:

$$\mathrm{Bi} = \frac{\text{resistance due to conduction}}{\text{resistance due to convection}} = \frac{L/k_{sol}}{1/h} = \frac{hL}{k_{sol}}.$$

The Nusselt number is approximately the ratio of plate length to thermal boundary layer thickness:

$$\mathrm{Nu} \simeq \frac{L}{\delta_T}.$$

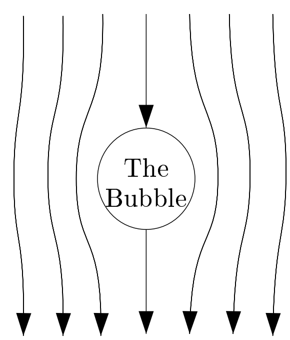
Since the heat transfer coefficient is roughly $h \simeq k_{fl}/\delta_T$, the boundary layer thickness is roughly $\delta_T \simeq k_{fl}/h$, so we can write:

$$\mathrm{Nu} = \frac{hL}{k_{fl}} \simeq \frac{L}{\delta_T}.$$

Thus despite their apparent similarity, the Nusselt and Biot numbers are thus very different animals, describing completely different ratios in different phenomena.

3. Floating Bubbles Out of Glass

(a) The bubble is floating up, so in its frame of reference the fluid is flowing down:



(b) This is just like problem set 8 #1, but slightly simpler and with a different constant:

$$F_b = F_d = fKA,$$

$$\frac{1}{6}\pi d^3 \rho g = \frac{16\mu}{\rho U d} \cdot \frac{1}{2}\rho U^2 \cdot \frac{1}{4}\pi d^2,$$

$$U = \frac{\rho g d^2}{12\mu}.$$

(c) The velocity which guarantees removal is the depth divided by the time. Solve for d and calculate, using that velocity:

$$d = \sqrt{\frac{12\mu U}{\rho g}} = \sqrt{\frac{12 \cdot 1.0\frac{\mathrm{kg}}{\mathrm{m\cdot s}} \cdot \frac{0.1\mathrm{m}}{60\mathrm{s}}}{3200\frac{\mathrm{kg}}{\mathrm{m}^3} \cdot 9.8\frac{\mathrm{m}}{\mathrm{s}^2}}} = 8.0 \times 10^{-4}\mathrm{m}.$$

(d) Since $V \propto d^2$, if we cut d in half, the velocity will be one quarter of what it was, so it will take four times as long to float out the bubbles 0.4mm in diameter.

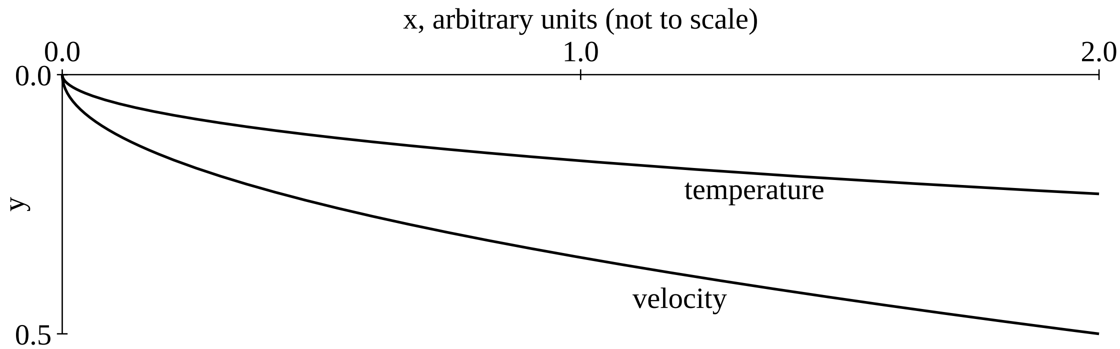
(e) The larger bubbles will have larger d and larger U and thus a larger Reynolds number, so if they're Stokes then both are:

$$\mathrm{Re} = \frac{\rho U d}{\mu} = \frac{3200\frac{\mathrm{kg}}{\mathrm{m}^3} \cdot \frac{0.1\mathrm{m}}{60\mathrm{s}} \cdot 8.0 \times 10^{-4}\mathrm{m}}{1.0\frac{\mathrm{kg}}{\mathrm{m \cdot s}}} = 4.3 \times 10^{-3}.$$

This is easily within the Stokes flow régime.

4. Estimating Fluid Velocity Using Heat Transfer

(a) With a Prandtl number of 10, the velocity boundary layer is about $\sqrt[3]{10} \simeq 2$ times thicker than the thermal boundary layer:



(b) To calculate the average heat transfer coefficient h_L:

- Calculate the Reynolds number from the free stream velocity and plate length: $\mathrm{Re}_L = \rho U_\infty L/\mu$.
- Calculate the Prantl number from the fluid properties: Pr$= \nu/\alpha = \mu c_p/k_{fl}$.
- Use a correlation to find the Nusselt number from the Reynolds and Prandtl numbers: $\mathrm{Nu}_L = f(\mathrm{Re}_L, \mathrm{Pr})$.
- The average heat transfer coefficient comes straight from the Nusselt number: $h_L = \mathrm{Nu}_L k_{fl}/L$.

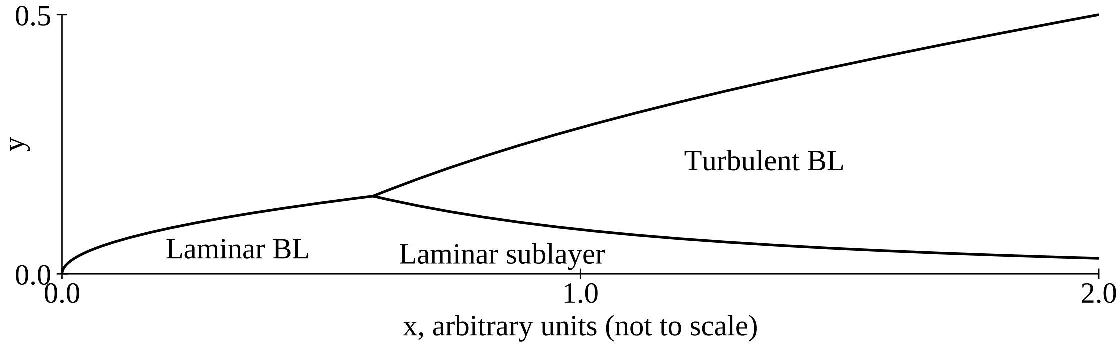
(c) The fourth assumption says that the solid conductivity is so high that in spite of large differences between the heat transfer coefficients at the plate leading and trailing edges, the temperature along the bottom of the plate is roughly uniform. If this is the case, then even a Biot-like number using the plate length would be small, so the Biot number using its thickness is extremely small. The plate temperature is pretty uniform everywhere, so $T_s \simeq T_0$.

(d) This is a bit of a spin on part 4b: you know the heat flux, and want to find h_L and thus U_∞. The steps should look like:

- Calculate the heat flux from the top of the plate to the air: $q = h_{air}(T_0 - T_{air})$.
- Using $T_s = T_0$, estimate the lower average heat transfer coefficient $h_L = q/(T_\infty - T_0)$.
- Calculate the Nusselt number from this heat transfer coefficient: $\mathrm{Nu}_L = h_L L/k_{fl}$.
- Reverse the Nusselt number correlation to calculate the Reynolds number from the Nusselt and Prandtl numbers: $\mathrm{Nu}_L = f(\mathrm{Re}_L, \mathrm{Pr}) \Rightarrow \mathrm{Re}_L = f'(\mathrm{Nu}_L, \mathrm{Pr})$. This is not hard for laminar flow (which the problem says you may assume), but might be if flow were turbulent.
- Calculate the free stream velocity from the Reynolds number: $U_\infty = \mathrm{Re}_L \mu/(\rho L)$.

5. Mass Transfer Boundary Layers

(a) This should have looked somewhat like:



(b) With a Prandtl number of 1000, the laminar boundary layer thickness ratio requested would be:

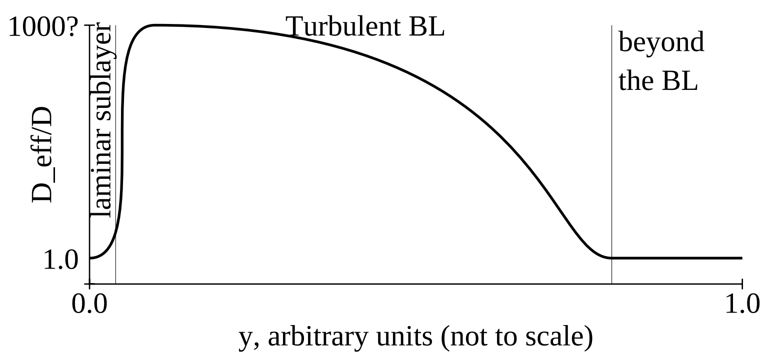
$$\frac{\delta_u}{\delta_C} = \frac{\sqrt[3]{\mathrm{Pr}}}{0.975} \simeq 10.$$

(c) The turbulent Prandtl number ν_t/D_t is about 1, so the turbulent diffusivity is approximately $D_t \simeq \mu_t/\rho$.

(d) The important things to show on your sketch are:

- the transition between laminar sublayer and turbulent layer near the plate (small y),
- the end of the turbulent boundary layer on top (large y),
- the diffusivity outside of the turbulent layer (in the laminar sublayer and outside the boundary layer) is the (low) molecular diffusivity,
- the effective diffusivity in the turbulent layer is much higher.

Based on these, your sketch should have looked something like (not to scale):



Part B: Open Book

1. Shear Stress in CD Injection Molding

(a) We need a general solution to:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = 0,$$
$$ru_r = A(\theta, z),$$
$$u_r = \frac{A(\theta, z)}{r},$$

where $A(\theta, z)$ is a kind of partial integration constant. We can reduce this to $A(z)$ by the assumption of axial symmetry.

(b) The average velocity is the flow rate divided by the cross section area, the latter of which is $2\pi r\delta$:

$$u_{av} = \frac{Q}{2\pi r\delta} = \frac{5\times 10^{-6}\frac{\text{m}^3}{\text{s}}}{2\pi\cdot 0.01\text{m}\cdot 0.001\text{m}} = 0.0796\frac{\text{m}}{\text{s}}.$$

Using this and the thickness gives a Reynolds number of

$$\text{Re} = \frac{\rho u_{av}\delta}{\mu} = \frac{1200\frac{\text{kg}}{\text{m}^3}\cdot 0.0796\frac{\text{m}}{\text{s}}\cdot 0.001\text{m}}{200\frac{\text{kg}}{\text{m}\cdot\text{s}}} = 4.8\times 10^{-4}.$$

This will produce *Stokes flow*, which means convection will be totally insignificant, and the $\rho u_r \partial u_r/\partial r$ term can be eliminated.

(c) The pressure field $p = f(r) - \rho g z$ tells us that the r-derivative is $\partial p/\partial r = f'(r)$, that is, it's not a function of z or θ. Based on this, the simplified r-momentum equation gives us:

$$0 = -f'(r) + \mu\frac{\partial^2 u_r}{\partial z^2},$$

$$u_r = \frac{f'(r)}{\mu}\frac{z^2}{2} + B(r,\theta)z + C(r,\theta),$$

where $B(r,\theta)$ and $C(r,\theta)$ can again be turned into $B(r)$ and $C(r)$ by the axial symmetry assumption.

(d) The solutions are equal to each other:

$$u_r = \frac{A(z)}{r} = \frac{f'(r)}{\mu}\frac{z^2}{2} + B(r)z + C(r).$$

This can work if $B(r)$, $C(r)$ and $f'(r)$ are replaced with B/r, C/r, and D/r respectively, where D/r is the pressure derivative $f'(r)$. The general result is:

$$u_r = \frac{D}{r}\frac{z^2}{2\mu} + \frac{Bz}{r} + \frac{C}{r}.$$

The problem didn't ask for this, but we can most easily fit the boundary conditions by choosing $z = 0$ halfway between the top and bottom of the mold, so that $B = 0$. With thickness δ, the boundary conditions are: $z = \pm\delta/2 \Rightarrow u_r = 0$; this plugs into the above solution as follows:

$$0 = \frac{D}{r}\frac{(\delta/2)^2}{2\mu} + \frac{C}{r},$$

$$C = -D\frac{(\delta/2)^2}{2\mu}.$$

This gives us an equation for u_r:

$$u_r = \frac{D}{2\mu r}\left(z^2 - \frac{\delta^2}{4}\right).$$

Finally, from the flow rate Q, which is the flow rate through a cylinder of radius r from $z = -\delta/2$ to $z = \delta/2$, we can get the last constant D:

$$\begin{aligned} Q &= \int_{z=-\delta/2}^{\delta/2} u_r 2\pi r dz \\ &= \int_{z=-\delta/2}^{\delta/2} \frac{D}{2\mu r}\left(z^2 - \frac{\delta^2}{4}\right) 2\pi r dz \\ &= \frac{\pi D}{\mu}\int_{z=-\delta/2}^{\delta/2}\left(z^2 - \frac{\delta^2}{4}\right) dz \end{aligned}$$

$$= \frac{\pi D}{\mu}\left(\frac{z^3}{3} - \frac{\delta^2}{4}z\right)_{z=-\delta/2}^{\delta/2}$$

$$= \frac{\pi D}{\mu}\left[\frac{2}{3}\left(\frac{\delta}{2}\right)^3 - \frac{\delta^3}{4}\right]$$

$$= -\frac{\pi D}{\mu}\frac{\delta^3}{6}.$$

$$D = -\frac{6\mu Q}{\pi\delta^3}.$$

At last, u_r is given by:

$$u_r = \frac{3Q}{\pi r\delta^3}\left(\frac{\delta^2}{4} - z^2\right).$$

The problem didn't ask again, but if you were interested, this gives you the pressure distribution:

$$f'(r) = D/r \Rightarrow f(r) = D\ln r + E = -\frac{3\mu Q}{\pi\delta^3}\ln r + E,$$

$$p = f(r) - \rho g z = -\frac{3\mu Q}{\pi\delta^3}\ln r - \rho g z + E,$$

where E is another integration constant.

(e) The most straightforward way to calculate the shear stress is to differentiate the above equation with respect to z. But as mentioned, the problem didn't require this.

It sufficed to note that u_r is parabolic in z, as shown in part 1c, with a maximum velocity related to average velocity by $u_{av} = \frac{2}{3}u_{max}$ as in channel flow. Again with z halfway between the plates, we can fit the parabola to the boundary conditions:

$$u_r = u_{max}\left(1 - \frac{z^2}{(\delta/2)^2}\right).$$

What we want is the shear stress, which comes from the derivative of u_r with respect to z:

$$\tau_{zr} = -\mu\frac{\partial u_r}{\partial z} = \mu u_{max}\frac{2z}{(\delta/2)^2} = \frac{8\mu u_{max} z}{\delta^2};$$

$$\tau_{zr}\big|_{z=\frac{\delta}{2}} = \frac{4\mu u_{max}}{\delta}.$$

At $r = 2$cm, $u_{av} = 0.0398$m/s (half the value of part 1b, since $u_r \propto 1/r$), so $u_{max} = 1.5u_{av} = 0.0597$m/s. Putting this and the process parameters into the above equation gives the shear stress:

$$\tau_{\text{zr}} = \frac{4 \cdot 200\frac{\text{kg}}{\text{m}\cdot\text{s}} \cdot 0.0597\frac{\text{m}}{\text{s}}}{0.001\text{m}} = 4.8 \times 10^4\,\frac{\text{N}}{\text{m}^2}.$$

Not such a big stress, but over many cycles can wear nickel features at micron lengthscales.

2. Ladle Metallurgy I: Scaleup of a Lab Design

Note: this 1/100-scale miniature ladle is really too small to work with: among other things, surface tension would play a huge role, and could make it hard to get any liquid metal into the tube at all. But the large difference in scale was necessary to illustrate the qualitative change in flow conditions from laminar and fully-developed to inviscid where Bernoulli applies.

(a) Since we have laminar fully-developed flow through a tube, what we want is the Hägen-Poisseuille equation plus gravity. Since ρg_z takes a similar place in the Navier-Stokes equations to $-\partial p/\partial z$, we can add it to $\Delta P/L$:

$$Q = \frac{\pi R^4}{8\mu}\left(\frac{\Delta P}{L} + \rho g\right).$$

(b) The pressure drop ΔP will just be $\rho g h_1$, and the tube length is h_2, so the flow rate will be:

$$Q = \frac{\pi R^4}{8\mu}\left(\frac{\rho g h_1}{h_2} + \rho g\right) = \frac{\pi R^4 \rho g}{8\mu}\left(\frac{h_1}{h_2} + 1\right),$$

$$Q = \frac{\pi \cdot (4 \times 10^{-4}\text{m})^4 \cdot 7000\frac{\text{kg}}{\text{m}^3} \cdot 9.8\frac{\text{m}}{\text{s}^2}}{8 \cdot 5 \times 10^{-3}\frac{\text{kg}}{\text{m}\cdot\text{s}}}\left(\frac{0.02\text{m}}{0.02\text{m}} + 1\right) = 2.76 \times 10^{-7}\frac{\text{m}^3}{\text{s}}.$$

This may seem miniscule in MKS units, but in cgs it's about 1/4 cm^3/second.

(c) To test for laminar flow, use the Reynolds number:

$$\text{Re} = \frac{\rho u_{av} d}{\mu} = \frac{2\rho R}{\mu}\frac{R^2 \rho g}{8\mu}\left(\frac{h_1}{h_2} + 1\right) = \frac{\rho^2 R^3 g}{4\mu^2}\left(\frac{h_1}{h_2} + 1\right);$$

$$\text{Re} = \frac{(7000\frac{\text{kg}}{\text{m}^3})^2 \cdot (4 \times 10^{-4}\text{m})^3 \cdot 9.8\frac{\text{m}}{\text{s}^2}}{4 \cdot (5 \times 10^{-3}\frac{\text{kg}}{\text{m}\cdot\text{s}})^2}\left(\frac{0.02\text{m}}{0.02\text{m}} + 1\right) = 614.$$

Flow is indeed laminar, and at this Reynolds number, the entrance length is about six times the diameter, or 5mm, so flow in 3/4 of the tube is beyond the entrance length and flow is fully developed.

(d) Now we have much larger velocity and tube diameter, so it's likely that the Bernoulli equation is what we need; results for just this geometry were derived in class (just points 1 and 4 from lecture are relevant here, which are top of the liquid steel and bottom of the tube):

$$(KE + P + PE)_{top} = (KE + P + PE)_{bottom}.$$

Taking $z = 0$ at the bottom of the tube, with $P = P_{atm}$ at both locations, gives:

$$\sim 0 + P_{atm} + \rho g(h_1 + h_2) = \frac{1}{2}\rho V^2 + P_{atm} + 0.$$

$$V = \sqrt{2g(h_1 + h_2)} = \sqrt{2 \cdot 9.8\frac{\text{m}}{\text{s}^2}(2\text{m} + 2\text{m})} = 8.85\frac{\text{m}}{\text{s}}.$$

The flow rate is then this times the cross section area ($5 \times 10^{-3}\text{m}^2$) which is $0.045\frac{\text{m}^3}{\text{s}}$, one cubic meter with seven tonnes of liquid steel every 20 seconds.

(e) At the tube exit, the Reynolds number is:

$$\text{Re}_x = \frac{\rho V x}{\mu} = \frac{7000\frac{\text{kg}}{\text{m}^3} \cdot 8.85\frac{\text{m}}{\text{s}} \cdot 2\text{m}}{5 \times 10^{-3}\frac{\text{kg}}{\text{m}\cdot\text{s}}} = 2.48 \times 10^7.$$

Flow is definitely turbulent, so use that correlation for the boundary layer thickness:

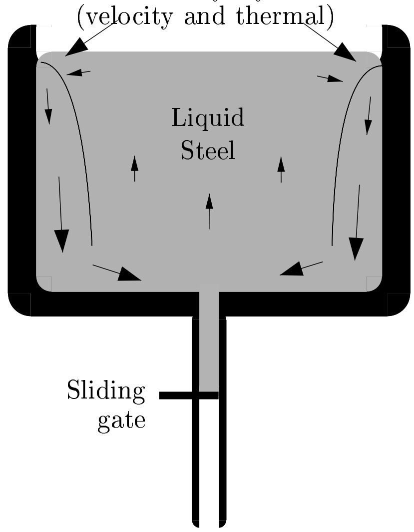
$$\frac{\delta_u}{x} = \frac{0.37}{\text{Re}_x^{0.2}} \Rightarrow \delta_u = \frac{0.37x}{\text{Re}_x^{0.2}} = \frac{0.37 \cdot 2\text{m}}{(2.48 \times 10^7)^{0.2}} = 0.025\text{m},$$

about 2.5 cm. This is just under 1/3 of the diameter, which is kinda pushing the limits of this methodology: we can't really say that most of the fluid has no turbulence or shear. There will therefore be more resistance to flow than Bernoulli allows, and somewhat lower flow rate than was calculated above.

(f) Nonetheless, the researchers who used laminar flow equations in this scaled-up case were grossly underestimating the resistance to flow.

3. Ladle Metallurgy II: Natural Convection

(a) The Prandtl number calculation should have preceeded any boundary layer sketching, as it would have shown that Pr< 1 so the velocity and thermal boundary layers are about the same size. In any case, your sketch should have looked something like:



(b) First the Grashof number, using the liquid metal height as the length scale:

$$\mathrm{Gr}_x = \frac{g\beta\Delta T x^3}{\nu^2} = \frac{9.8\frac{\mathrm{m}}{\mathrm{s}^2}\cdot 10^{-5}\mathrm{K}^{-1}\cdot 100\mathrm{K}\cdot(2\mathrm{m})^3\cdot(7000\frac{\mathrm{kg}}{\mathrm{m}^3})^2}{(0.005\frac{\mathrm{kg}}{\mathrm{m\cdot s}})^2} = 1.54\times 10^{11}.$$

The Prandtl number is:

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\mu c_p}{k} = \frac{0.005\frac{\mathrm{kg}}{\mathrm{m\cdot s}}\cdot 700\frac{\mathrm{J}}{\mathrm{kg\cdot K}}}{15\frac{\mathrm{W}}{\mathrm{m\cdot K}}} = 0.23.$$

The Grashof-Prandtl product is then 3.5×10^{10}, which is above 10^9 so flow is turbulent.

(c) The GrPr product puts this solidly in the $10^9 - 10^12$ régime where we can use the turbulent flow correlation in Adam Nolte's handout:

$$\mathrm{Nu}_L = 0.245\mathrm{Gr}_L^{2/5}\mathrm{Pr}^{7/15}(1+0.494\mathrm{Pr}^{2/3})^{-2/5} = 0.245\cdot 3.0\times 10^4\cdot 0.443 = 3237,$$

$$h_L = \frac{\mathrm{Nu}_L k}{L} = \frac{3237\cdot 15\frac{\mathrm{W}}{\mathrm{m\cdot K}}}{2\mathrm{m}} = 24300\frac{\mathrm{W}}{\mathrm{m}^2\cdot\mathrm{K}}.$$

This is a factor of sixty larger than a typical correlation for air, and even four times bigger than the 6000 number we used for h in water in test 1. Liquid metals are like that.

(d) With the boundary layer flowing down the sides of the ladle and to the bottom, when the gate is opened, the cooler metal from the boundary layers will exit first, followed by the hotter metal from the interior. That the flow rate down the boundary layers is similar to the flow rate through the nozzle means that this effect is quite significant, and as a result, thermal management in the downstream processes (tundish and continuous caster) can be very difficult.

4. This essay turned out to be harder than I thought because people had difficulty understanding what it was asking for. The opening process description was straightforward enough, but for the main "guide" step-by-step part, this was looking for an outline somewhat like those you wrote in problems 4b and 4d in Part A of the exam.