3.032 Practice Problem Set
Fall 2006
Solutions posted: End of lecture, 11/06/06

1. Your colleague is using the following viscoelastic model to predict a material's behavior: two Kelvin-Voigt models in series with each other and with a single spring. What is the effective stiffness of this arrangement at short time scales / high strain rates? At long time scales / low strain rates?

   SOLUTION: At high strain rates, dashpots act like rigid connections, and springs act like springs. At early times and high strain rates, the effective stiffness is therefore that of the sole spring; the two K-V elements are effectively rigid and do not displace.

   Conversely, at long times and low strain rates, dashpots act as though they are not even connected, so the effective stiffness is that of three springs in series.

2. Find the maximum difference (x%)in magnitude between the stress state that will produce plastic failure (yielding) of a metal according to the Tresca criterion or to the von Mises criterion. This is the location on the yield surface where the two surfaces are furthest apart.

   SOLUTION: The maximum difference between these two cases occurs for pure shear. On the yield surface diagram, this is the diagonal line that bisects quadrants 2 and 4. If we insert this stress state into the two criteria, we obtain different magnitudes of the shear stress magnitude σ_{13} that will be sufficient to cause failure by yielding.

   Tresca says: $\sigma_{13} = \sigma_y/2$; von Mises says: $\sigma_{13} = \sigma_y/\sqrt{3}$.

   Thus, the magnitudes differ by $\approx 13\%$, with Tresca being the more conservative criterion (i.e., lower stress required to meet the condition for failure by yielding).

3. Consider a material that is about to experience uniaxial loading of a magnitude sufficient to make the material yield. Could you prevent the material from yielding by applying loads on other axes? How? Why does this work?

   SOLUTION: See John Maloney's available on MIT Server/Resources for a nice discussion of Tresca related to this point. Recall that the yield criteria we covered relate the difference between maximum and minimum normal principal stresses (Tresca) or the difference among all three principal stresses (von Mises) to the magnitude of the yield stress. Intuition may tell you that adding more stress to a material will surely bring on failure, but if that stress decreases the difference between two principal stresses, it can actually prevent failure.

4. The stiffness matrix component C_{45} is shorthand for C_{2313} in the full stiffness tensor. However, the compliance matrix component S_{45} equals not S_{2313} but $4S_{2313}$. Show why by writing out ϵ_4 (using $\epsilon_i = S_{ij}\sigma_j$) and its equivalent $2\epsilon_{23}$ (using $\epsilon_{ij} = S_{ijkl}\sigma_{kl}$).

   Explain why this doesn't happen with the stiffness matrix.

   SOLUTION: By expanding the indicial notation,

   $$\varepsilon_4 = S_{41}\sigma_1 + S_{42}\sigma_2 + S_{43}\sigma_3 + S_{44}\sigma_4 + S_{45}\sigma_5 + S_{46}\sigma_6$$

Its equivalent, $2\varepsilon_{23}$, is expanded as

$$2\varepsilon_{23} = 2(S_{2311}\sigma_{11}+S_{2312}\sigma_{12}+S_{2313}\sigma_{13}+S_{2321}\sigma_{21}+S_{2322}\sigma_{22}+S_{2323}\sigma_{23}+S_{2331}\sigma_{31}+S_{2332}\sigma_{32}+S_{2333}\sigma_{33})$$

Using the fact that $\sigma_{ij} = \sigma_{ji}$, we note that the coefficients of the equivalent stress components σ_5 and σ_{13} are S_{45} and $4S_{2313}$, respectively. These coefficients must also be equivalent.

In general, $S_{ab} = S_{ijkl}$ for a and b both equal to 1, 2, or 3; $S_{ab} = 4S_{ijlk}$ for a and b both equal to 4, 5, and 6; and $S_{ab} = 2S_{ijkl}$ for all other cases (S_{14}, for example). In contrast, C_{ab} always equals C_{ijkl}, which can be seen by expanding $\sigma_a = C_{ab}\varepsilon_b$ and $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$.

As noted in Monday's review session, the 4 in the $4S_{2313}$ comes from two factors of 2: The first is from the fact that $\varepsilon_4 = 2\varepsilon_{23}$, due to the factor of two relationship between engineering shear strain ε_4 and tensorial shear strain ε_{23}. The second factor of 2 is from the fact that $S_{2313} = S_{2331}$, a fact you proved in PS5 using the four Kronecker deltas for C_{ijkl}, and you already know that $\sigma_{13} = \sigma_{31}$ from equilibrium. So when we write out $\varepsilon_{23} = S_{23kl}\sigma_{kl}$, we get $S_{2313}\sigma_{13} + S_{2331}\sigma_{31} = 2S_{2313}\sigma_{13}$.

5. Which of the following are easy slip plane / slip direction combinations for a copper crystal?

   (a) (111), [110]
   (b) (11-1), [00-1]
   (c) (101), [11-1]
   (d) (100), [0-11]

   SOLUTION: None of them. Remember that a slip system is a slip plane and a direction on that plane. Typically, the primary slip system is the close-packed plane and a close-packed direction on that plane. For Cu (fcc), this will be the {111}⟨110⟩ system. However, specific directions must be ON specific planes, which means that the dot product of the slip plane normal (which is oriented 90° to the slip plane) and the slip direction must be zero ($\cos(90^o)$ = 0).

   For a cubic system, it turns out that the slip plane normal has the same indices as the slip plane itself (here, the slip plane normal is ⟨111⟩). However, the dot product of some of the above combinations is not zero. For example, the first one would be:

   $$[111]dot[110] = (1 + 1 + 0)/(\sqrt{3}\,\sqrt{2}) = 2/\sqrt{6} \neq 0 \tag{1}$$

   Same deal for the second combination. The last two are also not the easy slip systems because, although the dot product is zero, these are not the close-packed planes of fcc.

6. What is the relationship (both direction and magnitude) between **b** and a for an fcc crystal? A bcc crystal?

   SOLUTION: For an fcc crystal, we covered this in lecture on 10.30.06. Looking at the (100) plane and assuming r is the atomic radius of a hard-sphere model of the fcc crystal, we see that

   $$4r = \sqrt{2a^2} \tag{2}$$

or $a = 2\sqrt{2}r$. We know that **b** is the vector that defines unit slip in the close-packed direction, so $\mathbf{b} = \frac{a}{2}\langle 110 \rangle = \frac{a}{\sqrt{2}}$.

For a bcc crystal, we know the atoms touch along the $\langle 111 \rangle$ or body-diagonal direction, so here $a = \frac{4r}{\sqrt{3}}$ and $\mathbf{b} = \frac{a}{2}\langle 111 \rangle = \frac{\sqrt{3}a}{2}$.

7. Your chemist friend is puzzled by a question of crystal symmetry: "A cubic material has the same stiffness in all three axes 1, 2, and 3, and an isotropic material has the same stiffness in all three axes 1, 2, and 3. So what's the difference?" How would you answer?

SOLUTION: One answer would be:

"You're right that the properties measured along three different axes may be numerically equal if the material is cubic or isotropic. However, that would be a special case in a cubic material, such that if I rotated the material and took measurements along a different three (yet still orthogonal) axes, the value in the second experiment need not equal the value obtained in the first.

In general, an isotropic material is an idealized case where the properties are independent of the direction in which they were measured, so it would not matter how I rotated a representative volume element of that material – I would ALWAYS measure the same value of a given property. A cubic material has a structure and properties that are defined by the orientation of the atoms. If the 1, 2, and 3 directions of my measurement are equal to the $\langle 100 \rangle$ vectors defining that unit cell, all the properties measured will be independent of direction. However, those that I measured along a rotated set of orthogonal vectors (e.g., $\langle 110 \rangle$) would not necessarily be equal to the values I measured along the $\langle 100 \rangle$.

So you see, my chemist friend, the properties of the cubic material are defined by the structure of the material (the arrangement of the atoms in the unit cell). The properties of the isotropic material are assumed independent of the arrangement of the atoms."