3.032 Problem Set 3
Fall 2006
Due: Start of lecture, 9/29/06

1. In PS2-4, you considered a femur under a torsional load that induced a spiral fracture, approximating the load as pure torsion.
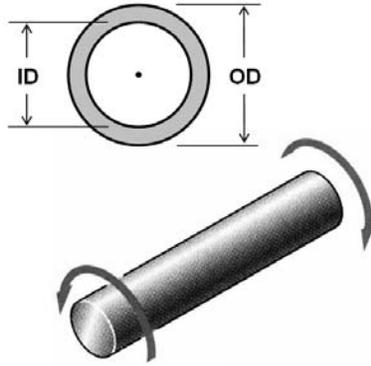
Figure 1: Femur under pure torsional loading T.

(a) Draw a representative in-plane element from the outermost surface of the femur diaphysis. The femur is idealized as an annular cylinder of uniform diameter subjected to a pure torque T (Figure 1). Indicate all the normal and shear stresses σ_{ij} acting on that material volume element, expressed only as a function of T.

Solution: $\tau_{12} = -\tau_o = Tr/J$. Using the values of r and J calculated in PS2-4, $\tau_o = (4.46$ $x\ 10^5/m^3)T$.
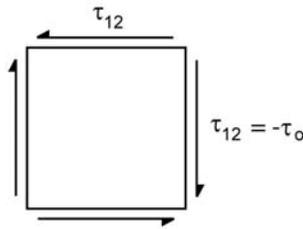
(b) Construct Mohr's circle of stress to identify the average normal stress σ_{avg} in the bone.

Solution: Since there is only a shear stress, the only point on Mohr's circle is the black spot shown at ($\sigma_{ii} = 0$, $\tau_{ij} = -\tau_o$), the magnitude of which is given in (a).

τ

$\tau_{max} = |\tau_o|$

$\sigma_{avg} = 0$

σ
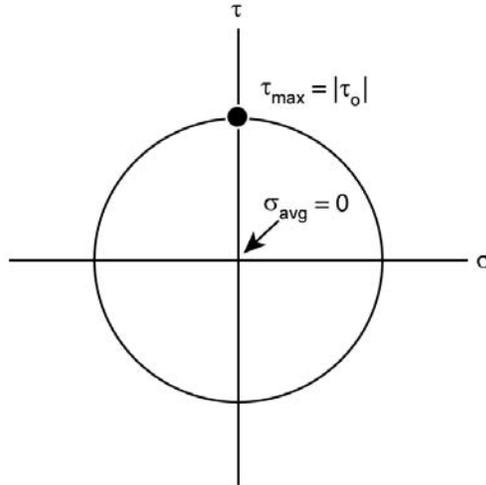
(c) Draw the orientation of the representative material element (indicate magnitude of θ_{ps}) and give the magnitudes of the principal normal stresses σ_1 and σ_2, assuming a critical torsional load F of 67 N for a moment arm of 2 m.

Solution: From the same Mohr's circle, it is clear that $2\theta_{ps} = 90°$, so $\theta_{ps} = 45°$ and the magnitudes of the principal stresses are $\sigma_1 = \tau$ and $\sigma_2 = -\tau$. The magnitude of τ_o is 60 MPa for this F and moment arm ($T = Fd$).

τ

$\tau_{max} = |\tau_o|$

σ_1

σ_2

σ

σ_1

σ_2

x' of principal stress

$45°$

x of pure shear case
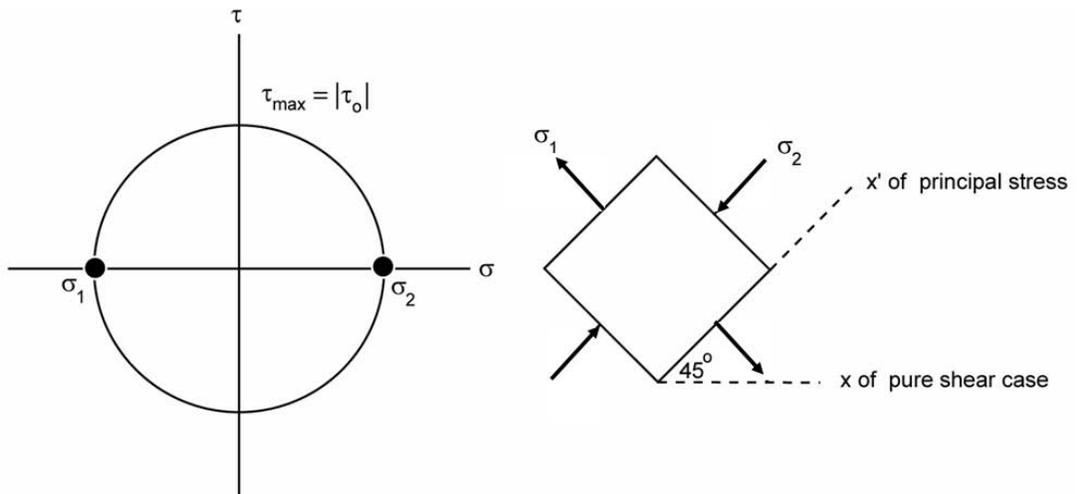
(d) Compare/contrast this finding with what is observed for the principal stress state and maximum shear state for the same bone subjected to uniaxial compression with a magnitude $\sigma_{11} = \sigma_1 = 60$ MPa.

Solution: In uniaxial compression of the same stress magnitude, the maximum shear stress is only $\sigma_1/2$, or 30 MPa. Interestingly, there is no plane of pure shear under uniaxial compression, though. You can see this by noting that the Mohr's circle is not centered at zero, but at $\sigma_1/2$.

2. Thin films used in microelectronics can develop considerable internal strain when they are deposited onto a substrate of a different material. The two materials may have different unit cell lattice parameters and/or may have different coefficients of thermal expansion, and the thin film is forced to take on the lateral dimensions of the substrate to which it is adhered. This strain can lead to substrate-film curvature (Figure 2). The midplane of such thin films is often approximated as being in a state of in-plane normal strain in cylindrical coordinates:

$$\epsilon_{r\theta} = 0; \epsilon_{rr} = \epsilon_{\theta\theta} = \epsilon_o - \kappa z$$

where ϵ_o is an isotropic extension of the substrate (and the film), κ is the curvature of the film/substrate, and z is the distance from the film neutral axis (its midplane).
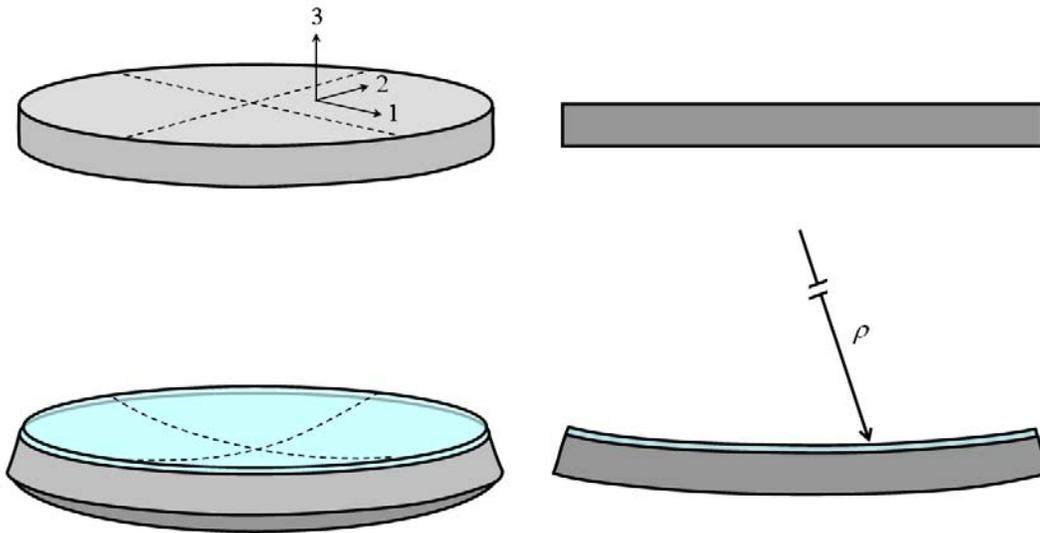
Figure 2: Top: Substrate before deposition; Bottom: Substrate and thin film after deposition (the actual curvature may be in the opposite direction depending on the mismatch between lattice parameters and thermal expansion coefficients).

(a) In a scientific programming language of your choice, write a program that can be used to determine the normal strain $\epsilon_{x'x'}$ and engineering shear strain $\epsilon_{x'y'}$ on the (thin film material) plane oriented at an angle θ with respect to your original coordinate system (x, y). Your program must report:

  i. $\epsilon_{x'x'}$ and engineering shear strain $\epsilon_{x'y'}$ for a given θ
  ii. principal strains ϵ_1 and ϵ_2 and the material volume element orientation θ_{pe}
  iii. maximum in-plane shear strain $\epsilon_{x'y',max}$ and the corresponding material volume element orientation $\theta_{pe,max}$
  iv. average normal strain ϵ_{avg}

Note: Mathematica, Matlab, Maple, C++, and Fortran90 are excellent choices to build this program. Practice with Mathematica will benefit you in future lab experiment analyses, so here's a good excuse to brush up (available on MIT Server). However, a brute-force execution in Excel is acceptable as a last resort. In all cases, you must provide the program and indicate the embedded equations with your problem set solution.

Solution: These equations were covered in lecture.

(b) Assume all the strain at the midplane of a Ge film deposited on a Si substrate comes from a mismatch of the unit cell lattice parameters a of those two materials. Compute that strain.

Solution: From www.matweb.com, the lattice parameters of Si and Ge can be obtained. $a_{Si} = 0.54$ nm; $a_{Ge} = 0.57$ nm. Thus, the extensional strain ϵ_o in Ge can be calculated from the difference in lattice parameters (engineering strain is the typical form):

$$\epsilon_o = (a_{Si} - a_{Ge})/a_{Ge} \tag{1}$$

which is -0.053 or 5.3% compressive strain.

(c) Is the film in a state of tensile or compressive strain, and will the film/substrate system show positive (smile) or negative (frown) curvature?

Solution: To place the film under compressive strain, a force must be applied at the interface that is equivalent to a compressive force applied to the midpoint of the film plus a negative moment. The curvature is therefore also negative.

(d) Assume the film is 500 nm in thickness and that you have measured the radius of curvature ρ of the film/substrate system as 10 km, the resolution of laser-based instruments used to measure wafer curvature. What is the in-plane normal strain $\epsilon_{rr} = \epsilon_{\theta\theta}$?

Solution:

$$\epsilon_{rr} = \epsilon_{\theta\theta} = \epsilon_o - \kappa z \tag{2}$$

The curvature component κz contributes only (250 nm)/(10 km) = 2.5 x 10^{-8} [unitless], so the strain in this case is due mostly to the lattice parameter mismatch (5.6%).

(e) This strain state is the principal strain state. How do you know this?

Solution: There are no shear strains present, so by definition this is the principal strain state.

(f) Assume a Cartesian coordinate system is a fair approximation in the plane strain state of the film (i.e., $\epsilon_{rr} = \epsilon_{\theta\theta} = \epsilon_1 = \epsilon_2$). Use your program to determine the orientation and magnitude of maximum in-plane shear strain in the Ge film.

Solution: Your program, using the equations of Mohr's circle for the plane strain case, would tell you that the maximum in plane shear strain is 0% because $\epsilon_1 = \epsilon_2$. However, Mohr's circle would tell you that the maximum shear strain exists in the 1-3 and 2-3 planes, and is equal to $\epsilon_{xy,max} = (1/2)5.3\% = 2.7\%$, at an orientation with respect to the original axes of 45°. This maximum shear strain is equal to the radius of Mohr's circle showing strain in all 3 axes.
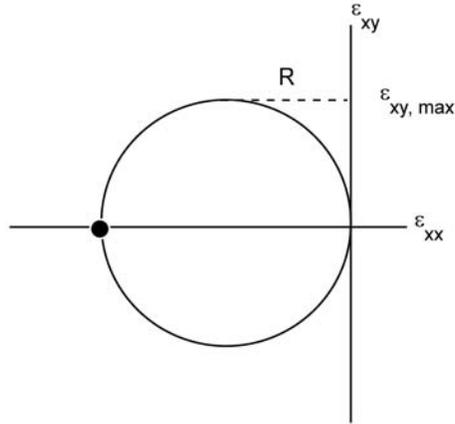
(g) If the principal strain state of a thicker piece of Ge were not in an equibiaxial strain state as described above, but instead a random biaxial strain state (e.g., $2\epsilon_1 = \epsilon_2$; $\epsilon_3 = 0$), use Mohr's circle to determine the magnitude of maximum shear strain and the plane of the film on which that shear strain would be (1–2, 1–3, or 2–3). You can assume the 3-direction to be through the film thickness.

Solution: The magnitude of the maximum shear is equal to the radius of Mohr's circle, or ϵ_1. This is on the plane represented by the two points at the diameter of this circle, or the 2-3 plane.
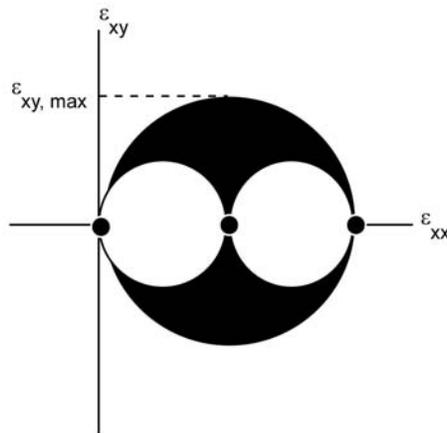
3. A beam of 2000 mm² cross-sectional area is composed of two pieces of wood glued together along a plane at an angle θ with the beam axis (Figure 3). It has been found from previous experiments that the joint will fail at 20 MPa and 10 MPa normal and shear stresses, respectively, and the wood at 56 MPa and 28 MPa normal and shear stresses, respectively. Determine the maximum allowable axial load that the bar can carry with a safety factor of two and the corresponding value of the angle θ.
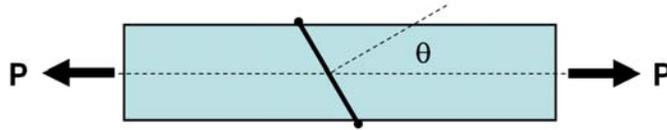
Figure 3: Wood pieces connected by glued joint.

Solution: The maximum load P and angle θ are found by solving the equations

$$\frac{2P}{A}\cos^2\theta = 20\,\text{MPa}$$

$$\frac{2P}{A}\cos\theta\sin\theta = 10\,\text{MPa}$$

to give P = 25 kN, θ = 26.6°.

Late addition: one of your colleagues came up with an elegant way to visualize the solution. To maximize the load, we seek the largest Mohr's circle that stays inside the σ = 20 MPa, τ = 10 MPa limitation. Since the load is uniaxial, the circle must pass through the point (0,0). The angle is calculated by trigonometry to be 26.6° and the maximum axial stress to be 25 MPa (resulting in a maximum load of 25 kN).
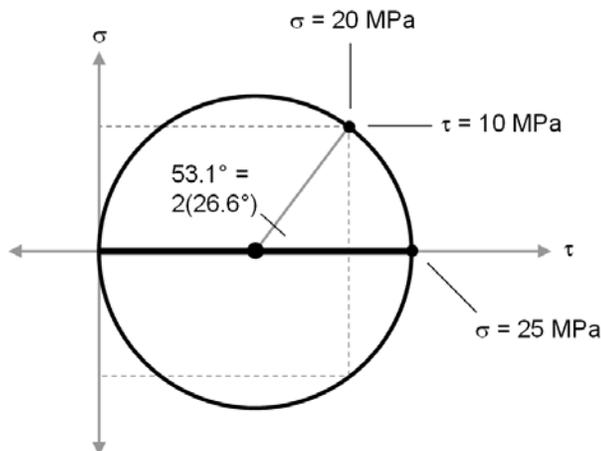
4. In 1654, the German scientist Otto von Guericke performed an experiment for Emperor Ferdinand III to demonstrate the nature of atmospheric pressure. Guernicke connected two copper bowls with an outer diameter D of 14 in, pumped out the air between them, and tried to pull them apart with two teams of eight horses each (Figure 4). The horses were unable to separate the bowls.

   (a) Assuming that Guernicke's pump was capable of removing 75% of the air from the assembly and idealizing the connected bowls as a sphere with a wall thickness of 0.4 cm, what was the maximum normal stress in the copper bowls before the horses started pulling?

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Figure 4: Otto von Guericke's experiment with two connected copper bowls, a vacuum pump, and two teams of horses [http://chem.ch.huji.ac.il/eugeniik/history/guericke.html].

Solution: The pressure on the sphere is 0.75 atm = 76 kPa. The maximum normal stress in the copper bowls is

$$\sigma = \frac{pr}{2t} = \frac{(-76\,\text{kPa})(0.178\,\text{m})}{2(0.004\,\text{m})} = 1.69\,\text{MPa (compressive)}$$

(b) Using a failure stress σ_F for copper in compression of 370 MPa, how thin in theory could the copper bowls be made without failure? If this value seems impractical, explain why using the terminology and concepts of 3.032.

Solution: The minimum thickness is

$$t = \frac{pr}{2\sigma_F} = 18.3\,\mu m$$

If the bowls were perfectly hemispherical and subject to no other loads than a uniform pressure, this vessel could in theory withstand a pressure differential of 0.75 atm. However, any deviations from sphericity (such as dents) or deflections from lateral forces would change the geometry (invalidating the pressure vessel assumptions), induce non-axial loads such as bending moments, and probably cause the bowls to buckle when vacuum is applied.

(c) Assume that the bowls are not hemispheres but rather two identical shells of revolution of $y = (7\,\text{in})\sin[x\pi/(14\,\text{in})]$ for x between 0 and 14 inches with axis of revolution $x = 7$ inches. The bowls are sealed at the joint with epoxy (Figure 5). The wall thickness is 0.4 cm and the interior is 75% evacuated as before. What is the actual shape of the stress distribution $\sigma(x)$, and what is its average value?

Solution: The shape of the bowls does not affect the total load at the joint, which is still $\pi r^2 p$. The only difference is that the bowls are joined at an angle of 114.6° (2 rad) rather than 180° as before:

$$2\theta = 2\tan^{-1}\left(\frac{d}{dx}\left[7\sin\left(\frac{\pi x}{14}\right)\right]_{x=0}\right) = 2\,\text{rad}$$
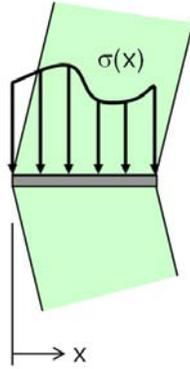
Figure 5: Epoxy joint between two shells of revolution.

The width of the joint is therefore

$$\frac{t}{\sin(1 \text{ rad})} = 4.75 \text{ mm}$$

The stress distribution could be approximated as constant with an average value of 1.42 MPa.