3.032 Problem Set 2
Fall 2006
Due: Start of lecture, 9/22/06

1. You make a questionable back-to-school purchase: a new pair of platform-sole sneakers that may make you look a bit taller. Much to your dismay, your "friend" tells you that when you put them on she can see that the green soles are compressed to about half their original height (i.e., half of the thickness shown in Figure 1(a)). Looking from the top, you can see that your soles were flush with the shoe upper before you put them on. However, under your weight, the soles stick out about 1 cm on each side.

3 cm
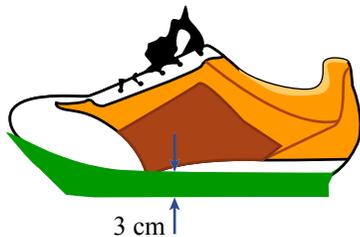
Figure by MIT OpenCourseWare.

1 cm

(a) buy-designer-shoes-online.ishoproute.com.jpg (b) www.istockphoto.com

Figure 1: (a) Sneaker before wearing, top view; (b) Your view of your sneakers while wearing.

(a) What is the Poisson's ratio of the material from which your sneaker soles are made?

Solution: To determine ν, we need the lateral strain ϵ_{lat} and normal strain ϵ_{norm}. The normal strain is given as $\epsilon_{norm} = -0.5$ (your sole compressed by half or 50% of its original height). The lateral strain must be calculated as:

$$\epsilon_{lat} = \Delta w / w_o$$

where w_o is the original width of your shoe. My shoe is 10 cm wide at the toebox, so I'll assume this as w_o. We are told your sole jutted out 1 cm on either side, so $\Delta w = 2$ cm and

$$\epsilon_{lat} = \frac{2\,cm}{10\,cm} = 0.2.$$

Then

$$\nu = -\frac{\epsilon_{lat}}{\epsilon_{norm}} = -\frac{0.2}{-0.5} = 0.4$$

1

(b) Based on this back-of-the-envelope calculation, what can you tell your friend about the likely composition and structure of that material?

Solution: Since metals and ceramics have $v \approx 0.3$ and polymers have $v \approx 0.4$–0.49, this is likely a sole made from a (green) polymer. In reality, sneaker soles were originally made of rubber, and in fact the term "sneaker" as synonym for athletic shoes is an American invention. It was supposedly coined by the US Rubber Company who made the first pair of Keds—the rubber sole made the wearer hard to hear when approaching on foot*. Although gum rubber sneakers still exist (think Chuck Taylor All-Stars), synthetic elastomers and polyurethane foams are now more common.
(*http://en.wikipedia.org /wiki/Sneaker_(footwear))

2. You are hanging from a rope that is of 4 cm diameter and originally of 6 m length. The rope is stationary and clamped tightly at the other end. Treat the rope as a square spring (i.e., Cartesian coordinates are OK) of effective spring constant k = 100 N/m.

   (a) At the clamped end, at the point halfway down the rope length, and at the end from which you're hanging, define all the components of the displacement vector $\mathbf{u} = (u_i, u_j, u_k)$.

   Solution: Let's assume you weigh 600 N, and that the clamped site is labeled A, the midpoint is labeled B, and the end on which you're hanging is labeled C. Since the rope has a stiffness k of 100 N/m, it will displace downward 0.01 m for every Newton of axial force applied. This means that the rope length will increase by (600 N)(0.01 m/N) = 6 m, for a final length of 6 + 6 = 12 m. However, the displacement is NOT uniform along the length of the rope, as shown in the figure below:
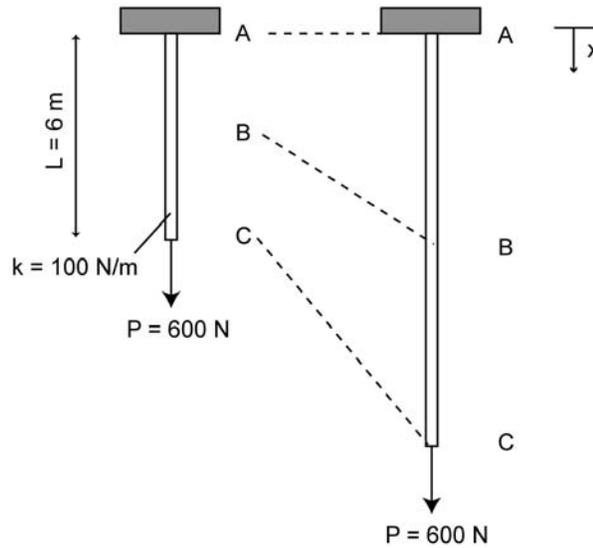
Thinking this through, $|u_A| = 0$ *m,* $|u_B| = 3$ *m, and* $|u_C| = 6$ *m. This means that* $u_x = u_1 = (P/k)(x/L)$, *where x is the distance from the clamped end and L is the initial rope length of 6 m. More generally,* $\mathbf{u} = (P/k)(x/L)\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$.

(b) Compute all the components of the small-strain, small-rotation strain tensor ϵ_{ij} at the end from which you are hanging.

Solution: By definition, ϵ_{ij} *is based on the partial derivatives of* $\mathbf{u}$:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_s}) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}) \\ \frac{1}{2}(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}.$$

Here, only u_1 *is nonzero and depends only on* x_1, *so only* ϵ_{11} *is nonzero and* ϵ_{ij} *becomes:*

$$\epsilon_{ij} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} = P/kL = 100\% & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) Compute the normal stress σ_{ii} at the same three points of the rope considered in (a).

Solution: At all points along the rope length,

$$\sigma_{ii} = \frac{F_i}{A_i} = \frac{600\,N}{\pi(2 \times 10^{-2}\,m)^2} = 4.8x10^5\,N/m^2 = 480\,kPa.$$

3

In other words, stress in a tensile member under an axial load is uniform along the member length.

(d) What do you notice about the variation in **u** and in σ_{ii} as a function of distance from the clamped end?

*Solution: Under this axial load, **u** varies along the length of the rope and is maximum at the end furthest from the clamp, while σ_{ii} remains constant. Here, the important point to notice is that the functional form of the displacement field may be quite different from that of the stress field in a material body.*

3. Concrete, sometimes called "the ultimate material," has enabled impressive achievements in construction for thousands of years. Concrete is also simple to make, consisting–in its simplest form–of only lime (calcium oxide), sand, gravel, and water. An unfortunate limitation of concrete, however, is that it is far, far weaker in tension than in compression. In this problem, you will develop a way around this limitation. The technique you will use, known as pretensioning, is commonly used to create long spans such as those pictured in Figure 3.

Image removed due to copyright restrictions. Please see
http://www.cement.org/bridges/truckee.htm

Figure 2: The Truckee River Bridge in Truckee, California—a 2002 Concrete Bridge Award Winner (Portland Cement Association, www.cement.org).

(a) Let us model a concrete span as a pin-and-roller-anchored beam with a length $L = 6$ m, a thickness $h = 0.8$ m, and width w (Figure 3(a)). The span is loaded with a distributed load q (force per *area*) of 50 kN/m^2. Determine the values and locations of the maximum tensile and compressive stresses in the beam. (It may be useful to construct shear and bending moment diagrams).

Solution: The stress in the beam is

$$\sigma = -\frac{My}{I} = -\frac{12My}{wh^3}$$

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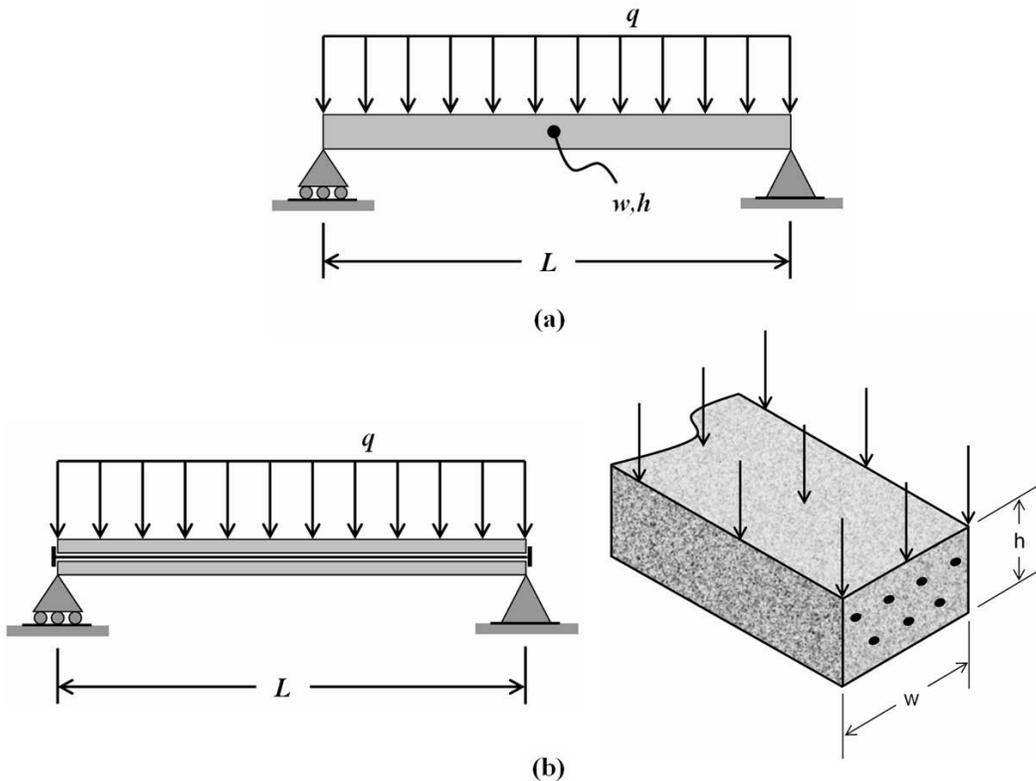
Figure 3: (a) Concrete span under distributed load q; (b) The same span with integrated steel rods to provide pretensioning.

where y is the distance from the neutral axis (in this case, the center) of the span. From free-body diagrams, the reaction force at each support is qwL/2 and the maximum bending moment is qwL²/8 at the center of the span. The maximum stress therefore has magnitude $3qL^2/4h^2$ = 2.11 MPa: negative (compressive) at the top of the span and positive (tensile) at the bottom.

(b) You have decided that it is best to allow no part of the concrete span to be in a state of tensile stress. To accomplish this goal, you will position steel bars along the length of the span before pouring the concrete. You will then stretch the bars, pour the concrete and allow it to harden, and release the tension (Figure 3(b)). The released steel bars will exert an axial compressive stress on the span that is superposed with the bending stresses. Determine the force per meter of span width to cancel the maximum tensile stress you calculated previously. Assume that the load created by the steel bars is distributed evenly across the cross section of the span and that the addition of the bars does not significantly change the cross section.

Solution: We require an axial compressive stress of $3qL^2/4h^2$ to cancel the tensile stress and the bottom of the span. The required force per unit width is $3qL^2/4h = 1.69\,MN/m$.

(c) You are told that the steel you are using has a failure strength $\sigma_F = 400\,\text{MPa}$ and the bars have a diameter of one inch. You wish to avoid failure in the bars with a factor of safety of two. What is the maximum load you can allow in each bar? How many bars are needed per meter of span width to provide the necessary compressive load that you calculated previously?

Solution: The maximum load P is

$$P = \frac{1}{2}\sigma_F A = \frac{\sigma_F \pi d^2}{8} = 101\,\text{kN}$$

The numbers of bars needed per meter of span width is

$$\frac{1.69\,\text{MN/m}}{101\,\text{kN}} = 16.7\,\text{m}^{-1}$$

4. A spiral fracture can occur when long bones such as the femur are exposed to an excessive torsional load (Figure 4(a,b)). The diaphysis (middle portion) of the femur consists of an outer circle of dense cortical bone surrounding a center of much weaker trabecular (meshlike) bone. Biomedical engineers have modeled the stronger cortical bone portion as an annulus, as shown in Figure 4(c). Let us assume that this annulus has an outer diameter of 26 mm and an inner diameter of 20 mm, and that the diaphysis has a length of 30 cm. The shear modulus of cortical bone is approximately 17 GPa.

   (a) The maximum shear stress occurs at the surface of the bone. If a shear stress $\tau_F = 60\,\text{MPa}$ will cause failure in cortical bone, find the maximum torsional load that this femur can withstand. What is this load in Newtons, assuming a lever arm of 2 m?

   Solution: The stress distribution under a torsional load T is

   $$\tau = \frac{Ty}{J}$$

   where y is the distance from the center and J is the polar moment of inertia. The maximum allowable torque is therefore

   $$T = \frac{\tau_F J}{(\text{OD}/2)} = \frac{\tau_F(\pi/2)\left[(\text{OD}/2)^4 - (\text{ID}/2)^4\right]}{(\text{OD}/2)} = 135\,\text{N–m}$$

   The load in Newtons corresponding to a lever arm of 2 m is 67.3 N.

(a)

(c)
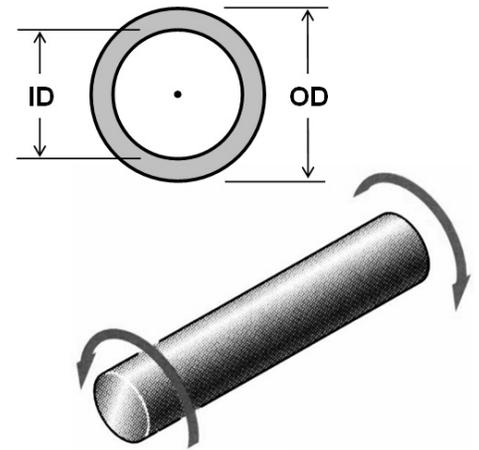
Courtesy Elsevier, Inc., http://www.sciencedirect.com. Used with permission.

Figure 4: (a) Fracture pattern of femur after excessive torsional loading [Vashishith et al., *J Orthopaed Res* 19:414 (2001)]; (b) X-ray of femoral spiral fracture [Ligier et al. *J Bone Joint Surg* 70B:74 (1988)]; (c) Loading configuration and annular model of cortical bone showing outer and inner diameters.

(b) If the dimensions of the cross section (outer and inner diameters) are reduced by 50%, by what percentage is the maximum torsional load reduced?

Solution: If the cross-section dimensions are reduced by half, the new maximum torque is calculated to be 16.8 N–m—an 88% decrease.

(c) What is the angle of twist at failure for the original cross section?

Solution: The angle of twist is calculated to be

$$\phi = \frac{TL}{JG} = 0.081 \text{ rad} = 4.7°$$