**3.032 Problem Set 1 solutions**
**Fall 2006**
**Due: Start of lecture, 9/15/06**

1. Thin-film silicon nitride cantilevers such as the one shown in Figure 1 are used in scanning probe microscopy, resonant frequency measurements, and electrostatic actuation. Let us approximate this cantilever as a clamped-free beam with a length of 200 μm, a width of 30 μm, and a thickness of 0.8 μm. Take the Young's modulus of thin-film silicon nitride to be 210 GN/m$^2$. The weight of the cantilever can be neglected.

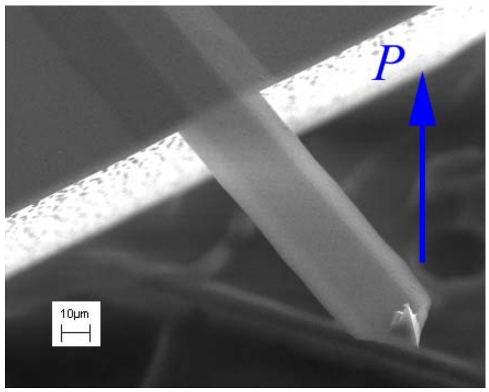



Image by Open Course Wares. Adapted from work by KristianMolhave, Wikimedia Commons.

Figure 1: Thin-film silicon nitride cantilever.

(a) If the free end of the beam contacts a surface (represented by a point load in Figure 1), and is deflected upwards by 1.0 μm, find the magnitude of the point load.

*Solution: The deflection of a clamped-free beam under a point load P at the end of the beam is*

$$\delta = \frac{PL^3}{3EI}$$

$$I = \frac{1}{12}wh^3 = 1.28 \times 10^{-24}\ \mathrm{m}^4$$

$$P = \frac{3EI\delta}{L^3} = 0.101\ \mu\mathrm{N}$$

(b) Draw a free-body diagram of the beam. Sketch the shear and bending moment in the beam along its length, labeling maximum and minimum values. Use the following sign convention for positive shear and bending moment [Beer and Johnson, *Mechanics of Materials* (1992)]:

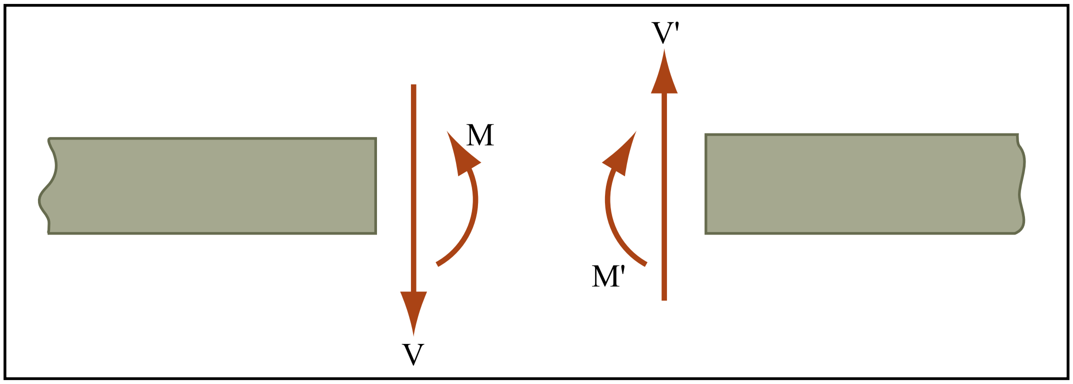
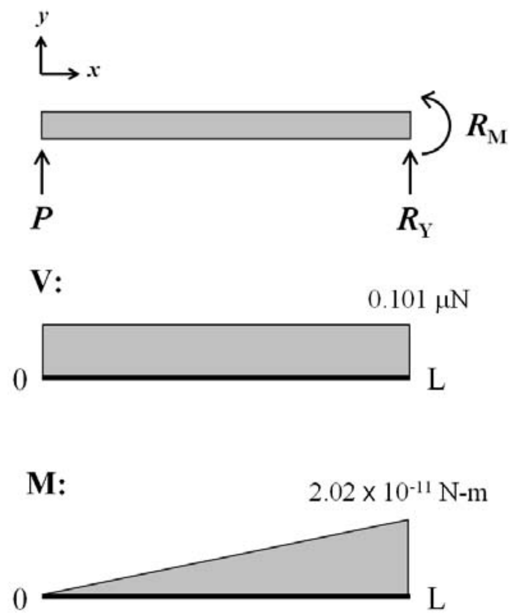



Figure by MIT OpenCourseWare.

*Solution:*




(c) In your scanning probe experiments, you need to be able to measure point loads that are only one-tenth of the value you calculated above. Assuming that a deflection of 1 μm produces a suitable signal for detection, how much longer does the cantilever need to be?

*Solution: The point load required to create a given deflection scales with $L^{-3}$, so the new length needs to be larger by a factor of $\sqrt[3]{10}$. The new length should be at least 431 $\mu$m.*

2. You are assembling a structure in a heavy wind, which acts as a distributed load $q$ (force per length) on one of the wide boards you are erecting (Figure 2). Your friend (weight 150 lb) can just barely keep the board upright by hanging from a flexible cord connected to the board. The board is connected to the ground by a pin joint, and the mass of the board is 30 kg/m. Ignore the effect of the wind on your friend.

   (a) Draw a free-body diagram of the board and find the wind load $q$ and the reaction forces at the pin joint.

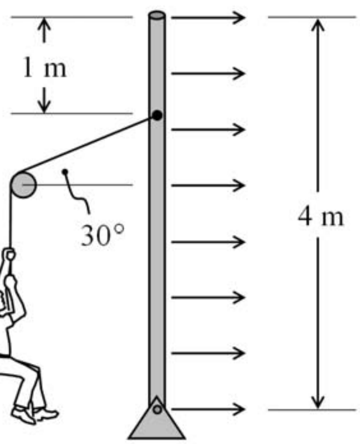



Figure 2: Upright board subject to a distributed load and a point load.

(b) Sketch the shear and bending moment in the board.

*Solution: The wind load q is found by summing the moments around the pin:*

$$T(3\,\mathrm{m})\cos 30^\circ - \frac{1}{2}q(4\,\mathrm{m})^2 = 0$$

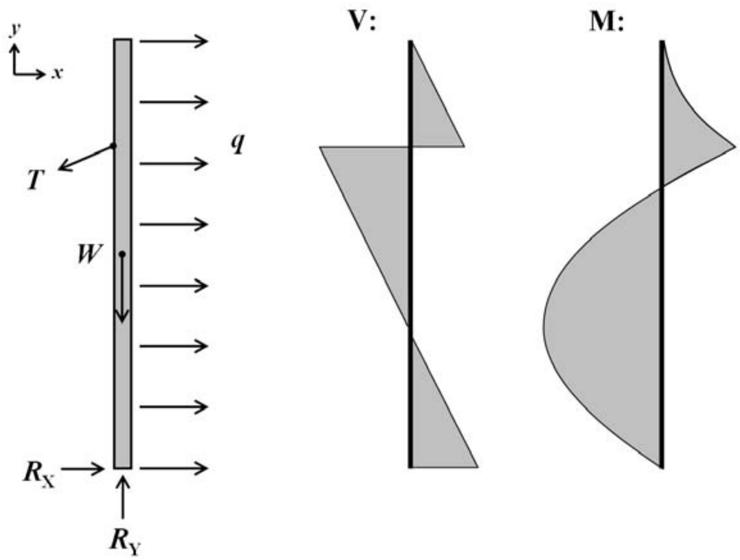
$$T = (150\,\mathrm{lb})(4.45\,\mathrm{N/lb}) = 667\,\mathrm{N}$$

$$q = 217\,\mathrm{N\,m^{-1}}$$

*The reaction forces are calculated as follows:*

$$R_\mathrm{Y} = T\sin 30^\circ + W = 1511\,\mathrm{N}$$

$$R_\mathrm{X} = T\cos 30^\circ - q(4\,\mathrm{m}) = -289\,\mathrm{N}$$



3. The Harvard Bridge on Mass. Ave. (Figure 3(a,b)) was rebuilt in the late 1980s due to the fact that the pin-and-hanger assemblies of the expansion joints on the bridge were the same as those of the Mianus River Bridge in Connecticut. The Mianus River Bridge collapsed in 1983 when a single pin became overloaded and caused the death of three people and serious injury of three others. Your task is to find the normal and shear forces and moment associated with that failure, which occurred at the position marked **x**, the midspan between pins C and D (Figure 3(c)).

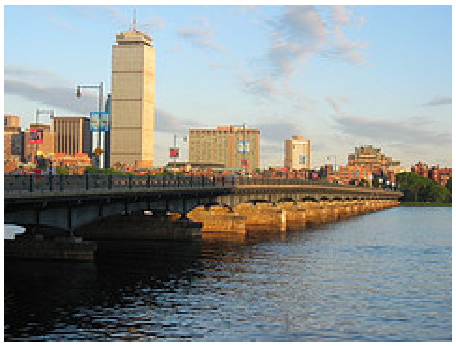
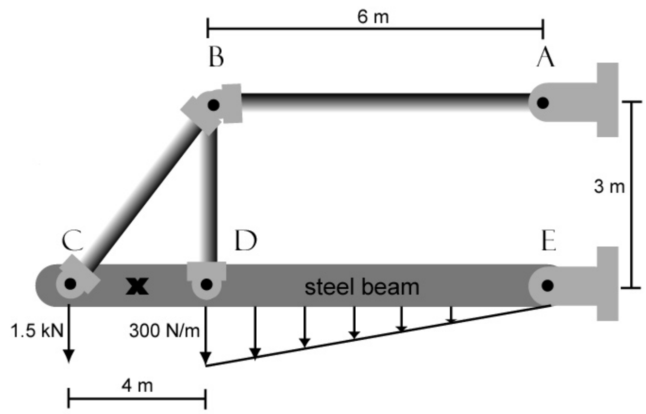


Image courtesy flickr user afagen.

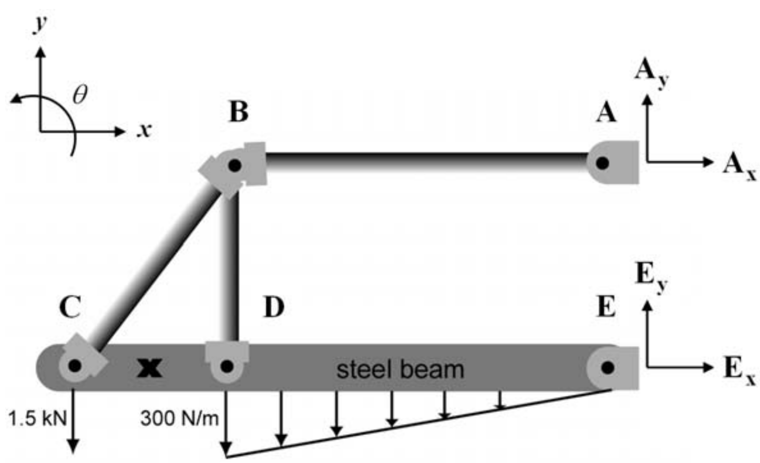



(c)

Figure 3: (a,b) Harvard Bridge; (c) Pin-and-hanger assembly, adapted from Hibbeler (2005).

(a) Draw a free body diagram of the entire pin-and-hanger assembly. The weight of the assembly can be neglected.

*Solution: The FBD is shown below. Note that the reaction moments are zero at pins A and E, as a pin cannot sustain a moment.*




(b) Determine the reaction forces about joint B.

*Solution: Members AB, BC, and BD are connected by pins and loaded at their ends only, so they can sustain only axial loads (this can be confirmed by FBD). From this consideration it is found that $A_y = 0$. Setting the sum of forces in the y direction in the complete FBD equal to zero:*

$$\Sigma F_y = 0 \rightarrow E_y - 1500\,\text{N} - 300\,\text{N}(6\,\text{m})(\frac{1}{2}) \rightarrow E_y = 2400\,N$$

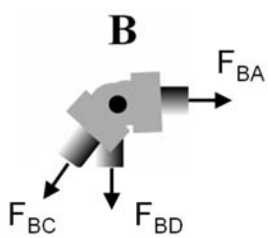
*Setting the sum of moments around point E equal to zero:*

$$\Sigma M = 0 \rightarrow (1500\,\text{N})(6\,\text{m}) + \int_0^6 (300\,\text{N})\left(\frac{x}{6\,\text{m}}\right)\text{dx} - A_x(3\,\text{m}) = 0$$

$$15000 + \left[100x^2\right]_0^6 - 3A_x = 0$$

$$A_x = \frac{18600}{3} = 6200\,\text{N}$$

*An FBD of pin B is also useful (note that the angle belongs to a 3-4-5 right triangle):*




*Setting the sum of the forces in the x direction at pin B to zero:*

$$\Sigma F_x = 0 \rightarrow -F_{\text{BC}}\left(\frac{4}{5}\right) + F_{\text{BA}} = 0$$

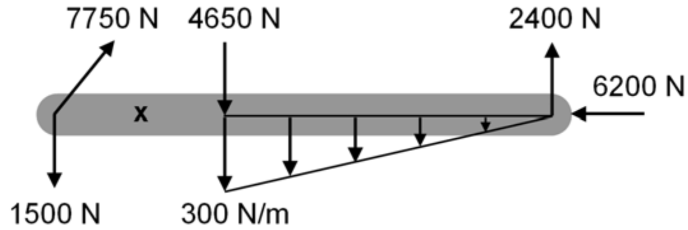
$$F_{\text{BC}} = \left(\frac{5}{4}\right)F_{\text{BA}} = \left(\frac{5}{4}\right)A_x = 7750\,\text{N}$$

*so member BC is under tension.*
*Setting the sum of the forces in the y direction to zero:*

$$\Sigma F_y = 0 \rightarrow -F_{\text{BC}}\left(\frac{3}{5}\right) - F_{\text{BD}} = 0$$

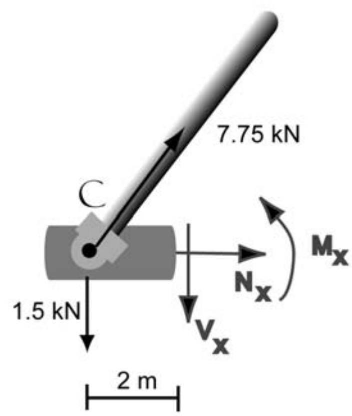
$$F_{\text{BD}} = \left(-\frac{3}{5}\right)F_{\text{BC}} = \left(-\frac{3}{4}\right)F_{\text{BA}} = -4650\,\text{N}$$

*so member BD in under compression.*
*The complete loading on the bottom beam is shown here:*




(c) Determine the forces along and normal to the horizontal beam, and the moment at the point of failure in the steel beam.

*Solution: Midway between pins C and D, we can apply the method of sections to the segment CX.*




*Demanding equilibrium,*

$$\Sigma F_x = 0 \rightarrow N_x = -7.75\,\text{kN} \cdot (4/5) = -6.2\,\text{kN}$$

$$\Sigma F_y = 0 \rightarrow V_x = 1.5\,\text{kN} - 7.75\,\text{kN} \cdot (3/5) = 3.15\,\text{kN}$$

$$\Sigma M_x = 0 \rightarrow M_x = (3/5)(7.75\,\text{kN})(2\,\text{m}) - (1.5\,\text{kN})(2\,\text{m}) = 6.3\,\text{kN-m}$$

4. Nanowires of amorphous silica ($SiO_2$) (Figure 4) are considered as possible optical waveguides in miniaturized electronics. For handling and assembly purposes, it is necessary to know and be able to measure the critical buckling load $P_{\text{cr}}$ of such nanowires.

   (a) For the above nanowire of 400 nm diameter and 1 mm length, determine the critical buckling load. Assume the value of the Young's modulus $E$ for bulk silica, which is 70 GN/m$^2$.

Fig. 4b in Tong, Limin, et al. "Subwavelength-Diameter Silica Wires for Low-Loss Optical Guiding. " *Nature* 426 (December 2003): 816-819.

Figure 4: Silica nanowire. (a) Amorphous silica fibers act as optical waveguides [Tong et al., *Nature* 426: 816 (2003)]. (b) Critical buckling load can be measured via compression with instruments such as atomic force microscopes.

*Solution: Assume that the bottom end of the wire in Figure 4(b) is fixed and the top end is free to move laterally. The critical load* $P_{\mathrm{cr}}$ *is then*

$$P_{\mathrm{cr}} = \pi^2 EI/(2L)^2 = \frac{\pi^2 E(\pi r^4/4)}{(2L)^2} = 2.17 \times 10^{-10}\,\mathrm{N} = 217\,\mathrm{pN} \tag{1}$$

*Other solutions are also acceptable if the end constraints are clearly stated. If the top end of the wire is assumed to be pinned, for example, the effective length will be* $0.7L$ *rather than* $2L$ *and the calculated critical load will be larger by a factor of approximately 8.2.*

(b) Using the cantilever in Problem 1, what is the deflection of that cantilever that you would need to achieve to impose this elastic instability?

*Solution: From that problem, the effective spring constant that defines the relationship between force P and deflection* $\delta = 3EI/L^3$*, using the E, I, and L of the silicon nitride cantilever. This value k = 0.101 N/m, so the cantilever would need to deflect an amount*

$$\delta = \frac{P_{\mathrm{cr}}}{k} = \frac{2.17 \times 10^{-10}\,\mathrm{N}}{0.101\,\mathrm{N/m}} = 2.15 \times 10^{-9}\,\mathrm{m} = 2.15\,\mathrm{nm}$$