Philosophy 244: #18—Sets, Actuality, Counterparts

Substantive Modal Theories

Start with the idea of a substantive theory of some kind of object—of quarks, natural numbers, or whatever. Theories of this sort are typically stated in nonmodal languages. Question: Is a nonmodal formulation good enough, or do some substantive theories lend themselves to, or even cry out for, modal elaboration?

That depends on the kind of object. Theories of quarks, probably not, because the principles of such a theory are not supposed to be necessary. But theories of numbers, or propositions, or sets, etc., are presumably supposed to hold necessarily. And there may be de re modal claims we want to make about the relevant objects as well. We'll focus here on the example of sets.

Kit Fine has done a lot to develop substantive modal theories (Fine [1980],Fine [1981],Fine [1982]).

Although: a counterfactual modality might be useful for distinguishing laws from accidental generalizations, or for talking about cause-effect relations.

ZF = Nonmodal Zermelo-Fraenkel Set Theory

X,Y,... range over sets, *x, y*..over anything. I am fudging the difference between ZF which is pure and ZFI which allows "urelements."

1) Extensionality
   $\forall z(z\epsilon X \leftrightarrow z\epsilon Y)\supset X=Y$

2) Union
   $\forall X\ \exists Y\ \forall z\ (z\epsilon Y\leftrightarrow\exists y\epsilon X\ z\epsilon y)$.

That is, $\forall X\ \exists Y\ Y = \cup X$

3) Power Set
   $\forall X\ \exists Y\ \forall Z\ (Z\epsilon Y\leftrightarrow Z\subseteq X)$

That is, $\forall X\ \exists Y\ Y = \{Z \mid Z\subseteq X\}$, that is, $\forall X\ \exists Y\ Y = \mathcal{P}(X)$

4) Infinity
   $\exists X((\exists Y\epsilon X\ \forall z\ z\notin Y)\ \&\ (\forall Z\epsilon X\ \exists Z'\epsilon X\ \forall w\ (w\epsilon Z' \leftrightarrow (w\epsilon Z \vee w=Z)))$

That is, there's a set containing the empty set and closed under $Z \rightsquigarrow Z\cup\{Z\}$

5) Regularity
   $\forall X\ (\exists Y\ Y\epsilon X \supset \exists Y\epsilon X\ Y\cap X = \varnothing)$

That is, each nonempty set has an ϵ-minimal member, that is, no set contains an infinite descending ϵ-chain .....$\epsilon\ x_n\ \epsilon....\epsilon\ x_3\ \epsilon\ x_2\ \epsilon\ x_1$

6) Separation (Schema)
   $\exists Y\forall x(\alpha[x]\supset x\epsilon Y)\supset \exists Z\forall x(\alpha[x] \leftrightarrow x\epsilon Z)$

That is, if α is bounded above by a set Y then there's a set $\{x \mid x\epsilon Y\ \&\ \alpha[x]\}$

7) Replacement (Schema)
   $\forall Z\ (\forall x\epsilon Z\ \exists! y\ \alpha(x,y) \supset \exists Y\ \forall x\epsilon Z\ \exists y\epsilon Y\ \alpha(x,y))$

That is, if Z is a set, then its range under any definable function is a set.

Where are null set and pairing axioms?

Fine's Proposed Modal Additions

A) $\Box$ZF
   $\Box\alpha$ for each axiom α of ZF

B) Rigidity of Sethood
   $\Box(Set(x)\supset\Box Set(x))$

$Set(x) =_{df} \exists Y\ Y=x$
Once a set, always a set.

C) Rigidity of Membership
   $\Box(x\epsilon y\supset\Box x\epsilon y)$

A set has its members necessarily.

D) Existence-Conditions
   $\Box\ (Set(x)\supset(Ex \leftrightarrow \forall y(y\epsilon x\supset Ey)))$

"A set has its members sufficiently."

Indefinite extensibility, modal structuralism, Hellman, Linnebo.

Actuality and Multiple Indexing

The kind of modal language we've been working with has some expressive gaps. How are we to represent, for instance,

(1) It might have been that everyone (actually) happy was unhappy

First try, where $\varphi(x)$ means that x is happy.

(2) $\Diamond\forall x(\varphi(x)\supset\neg\varphi(x))$

That's clearly not right!

(3) $\forall x(\varphi(x)\supset\Diamond\neg\varphi(x))$.

This is closer, but it misses that the happy people could've been unhappy *together*. If we had an actuality operator A, we could say

(4) $\Diamond\forall x(A\varphi(x)\supset\neg\varphi(x)))$

Another way to express this is with plural quantifiers:

(4*) $\exists xx\ [(\forall x\ (x \text{ is an } xx \leftrightarrow \varphi(x))\ \&\ \exists yy\ (yy = xx\ \&\ \Diamond\ \forall z\ (z \text{ is a } yy \supset \varphi(z)))]$

Now we need to give a semantics for A. Spose we're evaluating (5) in the actual world w. It's true in w iff there's a world u where

(5) $\forall x(A\varphi(x)\supset\neg\varphi(x))$

is true. Well, what does that take? An object in u should satisfy $A\varphi(x)$ there just in case it's *actually* happy, that is, happy back in w. Truth-value assignments will therefore have to be relative to two worlds or more generally two "indices": the evaluation index, in this case u, and the reference index, in this case w. Except where actuality is concerned, the reference index w_2 is just along for the ride:

That we've got two indices here means that we're doing so-called "two-dimensional semantics."

($V\varphi2$) $V_\mu(\varphi(x),<w_1,w_2>)=1$ iff $<\mu(x),w_1>\epsilon V(\varphi)$
($V\Diamond2$) $V_\mu(\Diamond\alpha,<w_1,w_2>)=1$ iff $\exists w_3$ s.t. w_1Rw_3 and $V_\mu(\alpha,<w_3,w_2>)=1$
(VA2) $V_\mu(A\alpha,<w_1,w_2>)=1$ iff $V_\mu(\alpha,<w_2,w_2>)=1$

So the actuality operator turns the evaluation world into the reference world. Let's see how this works in the case at hand.

$V_\mu(\Diamond\forall x(A\varphi(x)\supset\neg\varphi(x)),<w_1,w_1>)=1$ iff
for some w_2 visible from w_1 $V_\mu(\forall x(A\varphi(x)\supset\neg\varphi(x)),<w_2,w_1>)=1$ iff
for every x-variant ρ of μ, $V_\rho(A\varphi(x)\supset\neg\varphi(x)),<w_2,w_1>)=1$ iff
for every x-variant ρ of μ, if $V_\rho(A\varphi(x),<w_2,w_1>)=1$ then $V_\rho(\neg\varphi(x),<w_2,w_1>)=1$ iff
for all x-variants ρ..., if $V_\rho(\varphi(x),<w_1,w_1>)=1$ then $V_\rho(\neg\varphi(x),<w_2,w_1>)=1$, iff
for all worlds u, if $<u,w_1>\epsilon V(\varphi)$ then $<u,w_2>\notin V(\varphi)$

which is what we wanted. All of this actually makes best sense against the background of a slightly different form of modal semantics than we have been using, which let me now explain.

Real-World Validity

Our current frames are <WR>, and our models <WRDV>. An alternative approach championed by (among others) Kripke is to let frames be <W,@,R> with @ the actual world; models are expanded to include @ to. Against the background of this kind of model the second or "reference"-world in the evaluation rules can always be considered the actual world of the model.

Validity is now defined as truth in the @-world of every model. Davies and Humberstone in their classic paper on the subject call this "real-world" validity. This is philosophically preferable because it lessens the danger of confusing necessity, which properly understood involves quantification over worlds not models, with validity or logical truth, which involves quantification over models not worlds.

Davies and Humberstone [1980]

An example of a real-world valid formula that isn't generally valid, that is, true in all worlds of every model, is $\alpha\supset A\alpha$. .

Axioms for this semantics (we'll stick to the propositional version) are S5 with the usual rules plus:

(A1) $A(\alpha\supset\beta)\supset(A\alpha\supset A\beta)$
(A2) $A\alpha\leftrightarrow\neg A\neg\alpha$
(A3) $\Box\alpha\supset A\alpha$
(A4) $A\alpha\supset\Box A\alpha$

The last of these may look wrong. If α is a contingent truth, then the fact that actually-α may seem equally contingent. So $\Box$ is covering too much. Also $\Box$ is covering too little. Because $\alpha \supset A\alpha$ seems like it is some sense necessary, but $\Box(\alpha \supset A\alpha)$ can't be inferred from the axioms; (A3) comes closest. What's going on? Our intuitions here are tracking a second notion of necessity/contingency.

α is necessary$_1$ iff α would have been true no matter how things had been different — = no matter which w *had* been actual
α is necessary$_2$ iff α is true no matter how things really are — = no matter which w *is* actual

(A4) is necessary in the first sense but not the second. $\alpha \supset A\alpha$ is necessary in the second sense but not the first.

Which is expressed by $\Box$? Clearly necessity$_1$. To express the second D & H introduce a "fixedly" operator $\mathcal{F}$; a sentence α is necessary$_2$ iff $\mathcal{F}A\alpha$. A rough intuitive reading of $\mathcal{F}$ is: from the perspective of any world, conceived as actual. The semantics is simplicity itself. Say that two models M and M* are variants of one another iff they differ only in which world is nominated as actual. Then

(V$\mathcal{F}$) $V(\mathcal{F}\alpha, <@,@>)=1$ iff for all variants M* of M, $V(\alpha, <@^*,@^*>)=1$

This relates to Gareth Evans's distinction between "deep" and "superficial" necessity and contingency, drawn in response to Kripke's examples of contingent a priori statements like "this stick is one meter long." A similar example is $A\alpha \supset \alpha$. These are superficially contingent in that their necessitations$_1$ $\Box\alpha$ are false. But they are deeply necessary because their necessitaitons$_2$ are true: $\mathcal{F}A(A\alpha \supset \alpha)$.

Counterpart Theory

Our original models of QML were constant domain models. Then we generalized to expanding domain models and allowed finally arbitrary variation among domains. Now we go to the opposite extreme: rather than insisting that domains *overlap* to such and such an extent, we insist that they're completely disjoint. A particular reason for being interested in this is that if you're a "modal realist" a la David Lewis, then it's hard to believe there *could* be overlap.

If the domains are to be disjoint, something all have to be done about the evaluation rules. Here's why; suppose we express the fact that x is essentially φ by saying that $\Box(Ex \supset \varphi(x))$. Then since x exists in a single world only, it will wind up having all its properties essentially. $V_\mu(\varphi(x) \supset \Box(Ex \supset \varphi(x)), w)=1$.

There are passages in Leibniz that suggest he was prepared to draw this "superessentialist" inference. If Adam is such that Peter denies Christ many generations later, then that is essential to him. It pertains to your essence to be born on a certain day. But there are other passages where he says: look, there are people very *like* Adam – *other* Adams he calls them – in other worlds who lack this property. If we analyze what it is for Adam to have a property essentially in terms of the behavior of these *other* Adams, then the superessentialism no longer follows.

This is what Lewis does in effect. The "other Adams" he calls Adam's *counterparts*. And what it is for Adam to be essentially φ is for all his otherworldly counterparts to be φ. This may seem strange but we're arguably up to our neck in it already. The possibility of flying pigs in our world already hinges on goings on in other worlds. Why should the possibility for me of being later to class not similarly hinge on, or be witnessed by, the features of other people–people I am not but could have been.

The mechanics are as follows. Models will be 6-tuples <WRDQCV>, where Q maps worlds to their (disjoint!) domains , all subsets of D, and C is a binary relation on D.) And now the rule for $\Box$ is (assuming that R is universal):

C is like accessibility, but for objects. Just as counterfactual worlds witness possibilities for our world, counterfactual doubles witness possibilities for our world's inhabitants.

(V$\Box$') $V_\mu(\Box\alpha, w)=1$ iff for all *x*-variants ρ of μ taking x to a counterpart in u, $V_\rho(\alpha, u)=1$.

Under certain conditions this gives essentially the same results as ordinary QML with the Barcan Formula. The conditions are that R is an equivalence relation and everything has a unique counterpart in every world. Because what you can then do is convert your counterpart model into a regular model by letting the elements of some one world w* stand in for their counterparts in other worlds.

That is, given <WRDQCV>, form <W'R'D'V'> as follows. Let W'=W, R'=R, D'=D_{w*}, and for $y_i \epsilon$D', <y_1,....,y_n,w>ϵV'(φ) iff, where z_i is y_i's unique counterpart in w, and <z_1....,z_n,w>ϵV(φ). One can also do it backwards, taking <W'R'D'V'> to <WRDQCV>. The domain of w = {<u,w> | uϵD'}, and xCy iff their first element u is the same.

"What this means is that we take the individuals of the new model to be the members of some selected world w*, and stipulate that they are to do in another world w what their counterparts in w do in the original model."

So it begins to look as though "counterpart theory is just ordinary modal predicate logic in disguise." But no, because C need not be an equivalence relation s.t. everything has just one counterpart in every world. Indeed you wouldn't expect it to be, says Lewis, if counterparthood is a kind of similarity relation; y is x's counterpart if it's similar enough to x and nothing in its world is more similar. Don't expect transitivity because z might be similar enough to y, and y to x, without z being similar enough to x. Don't expect symmetry because y might be the most similar thing over there to x here without x being the most similar thing to y over here. Don't expect uniqueness because there could be ties.

This can help with modal sorites paradoxes.

This can help with fission cases.

This has enormous effects on the logic, as H & C observe. Even some very simple K-theorems are now invalidated. For instance

(1) $\Box(\varphi(x)\&\forall x\varphi(x))\supset\Box\forall x\varphi(x))$.

Spose W = {w_1, w_2, R=W×W, D={y_1, y_2}, Q(w_i)={y_i}, C is identity, and V(φ)={y_1}. Then when μ(x)=y_1, the antecedent is true – just because there are no counterparts to x but itself – but the consequent is false, since φ is not true of y_2.

Contingent Identity

The physicalist wants to say that I am identical to my body B. How can she though, when I have a property B lacks: I could have existed apart from B (say as B'), while B couldn't. If I am my body, how can it be that:

(1) I and my body are such that the first might have existed apart from the second.
(2) My body and itself are not such that etc.

The answer is given by these counterpart-theoretic translations:

(1C) There is a world w containing a unique personal counterpart X of me, and a unique bodily counterpart Y of me, and X sits in w apart from Y

(2C) There is no world w containing a unique bodily counterpart of me X, and a unique bodily counterpart of me Y, such that X sits in w apart from Y.

The distinctive contribution of the different referring terms–*Me*, *Body*—is not semantic is not semantic; they do change the object under discussion. They function rather as an element of *context* that makes one or another counterpart relation salient. Putting *Body* for *Me* thereby changes *what is being said about the self-same object.* "...can't exist without Body" expresses one property coming after a person-type name for SJY, another coming after a body-type name for SJY. The idea goes back to Quine; *Danielito is so-called because of his size* attributes a different property to the referent than *Uncle Daniel is so-called because of his size.*

The Crimmins-Perry account of belief-reports is built on the same idea. *Lois believes Superman wears a cape* attributes to Lois the property of believing x to be exciting via her Superman-idea.

References

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