Philosophy 244: 244 #3—Basic Modal Systems

Time to start looking at some specific logical systems, starting with our "base" system K – the system such that theoremhood in it corresponds to "absolute" validity, ie. validity in all seating arrangements whatsoever. Logical systems can be presented in a number of ways, but we'll do it axiomatically. This involves specifying

(a) the language in which wffs are formulated
(b) a selected set of wffs known as the axioms
(c) a set of transformation (inference) rules.

The system's *theorems* will be the closure of the axioms under the inference rules, ie., everything you can get from the former by repeated application of the latter.

Crucially all of the notions (a), (b), and (c) have got to be *effective* in the sense that there's a mechanical procedure for determining what counts as a wff of the language, what counts as an axiom, and what counts as an (instance of) an inference rule.

These procedures have also got to be non-semantic in nature. One needn't know what anything stands for or what is true to figure out what the wffs, axioms, and permissible inferences, are. For this reason the axiomatic approach to logical systems is sometimes called the (or a) syntactic approach; the intended contrast being with the semantical approach via models and validity.

System K

Earlier we talked about absolute validity – validity in all seating arrangements, or all frames – and said that this was the notion of validity captured by system *K*. What is meant by this? That the theorems of *K* are precisely and without exception the absolutely valid wffs. Another word for absolute validity is thus *K*-validity.

To present *K* I've got to tell you (a) what the wffs are, (b) what the axioms are, and (c) what the inference rules are.

(a) The wffs, and indeed the wffs of all systems to be considered for quite a while, are just the wffs of modal propositional calculus.

(b) The axioms are of two kinds; *PC* as written is a "schema" that lays out a whole bunch of axioms at once.

PC If α is a PC-valid wff, then α is an axiom.

K $\Box(p \supset q) \supset (\Box p \supset \Box q)$ is an axiom.

Note that *PC* stands in for an *infinite number* of axioms. How does that fit with the idea that the axioms have to be specified effectively or algorithmically? *K* is just one particular sentence, built on the atomic sentences *p* and *q*. Anything of *K*'s form will be a *theorem*, only the others are arrived at using an inference rule.

(c) *K* has three rules:

US Substitution: Replacing $p_1..p_n$ in a theorem by $\alpha_1..\alpha_n$ yields a theorem.
MP or Modus Ponens or Detachment: If α and $\alpha \supset \beta$ are theorems, so is β.
N or Necessitation: If α is a theorem, so is $\Box\alpha$.

To represent the result of systematically substituting the β_is for the p_is in β, we write $\alpha[\beta_1/p_1,...,\beta_n/p_n]$. Where α is a wff and S is an axiomatic system, we write $\vdash_S \alpha$ to express that α is theorem of S. When $\vdash\alpha$ is derivable from $\vdash\alpha$ in the going modal system we write $\vdash\alpha \rightarrow \vdash\beta$. With this we can state the rules more simply as

$US \vdash \alpha \to \alpha[\beta_1/p_1,...,\beta_n/p_n]$.

$MP \vdash \alpha, \vdash \alpha \supset \beta \to \vdash \beta$

$N \vdash \alpha \to \vdash \Box\alpha$

Notice that only the third of these is a specifically *modal* rule. N may strike you as funny; why should α suddenly be entitled to sprout a $\Box$ at the front? Remember that α is only a theorem if it is *valid*; if α is valid, meaning something like true in virtue of the meaning of its logical symbols alone, then it could hardly fail (it seems) to be true in all possible worlds in virtue of those same meanings, which means its necessitation is valid. Later when we get to actuality we'll see a possible hold in this reasoning. (*The President = the actual President* is true in virtue of the meaning of its logical symbols, but it doesn't hold in all worlds.)

Proofs and Provability

A proof in S is a finite sequence of wffs, each of which is either (i) an axiom of S or (ii) derived from earlier wffs by (iia) applying one of the transformation rules or (iib) making a definitional substitution. Each line of a proof contains three items:

  a wff,
  a reference number for that wff, written immediately before it
  a justification for the wff, written on the far right (not the far left as in the book)

Justifications are basically explanations of how it is that the wff satisfies condition (i) or condition (ii). As for (i), axioms falling under PC are justified by writing "PC," perhaps with a number to indicate which of the sample PC theorems the wff happens to be. Axioms falling under K (there's only one of these!) are justified by writing "K."

As for (ii), you should list both the inference rule (or definition) involved and the lines that served as premises. Applications of US are justified by writing the earlier line number and then indicating with the square bracket notation the substitution that was made. Applications of MP and N are marked by "?MP" and "?N" respectively. So, let's do it. Like the book we'll first prove two theorems in full detail, and then give some methods for abbreviating proofs. *K1* and *K2* are proved on p. 27; we'll do theorem *K1*.

K1	$\Box(p\&q) \supset \Box p\&\Box q$	
1.	$p\&q \supset p$	PC1
2.	$\Box(p\&q \supset p)$	(1)×N
3.	$\Box(p \supset q) \supset (\Box p \supset \Box q)$	K
4.	$\Box(p\&q \supset p) \supset (\Box p\&q \supset \Box p)$	(3),[$p\&q/p, p/q$]
5.	$\Box p\&q \supset \Box p$	(2),(4)×MP
6.	$p\&q \supset q$	PC2
7.	$\Box(p\&q \supset q)$	(5)×N
8.	$\Box(p\&q \supset q) \supset (\Box p\&q \supset \Box q)$	(3),[$p\&q/p$]
9.	$\Box p\&q \supset \Box q$	(7),(8)×MP
10.	$(p \supset q) \supset ((p \supset r) \supset (p \supset (q\&r)))$	PC3
11.	$(\Box(p\&q) \supset \Box p) \supset ((\Box(p\&q) \supset \Box q) \supset (\Box(p\&q) \supset (\Box p\&\Box q)))$	(10), [$\Box(p\&q)/p, \Box p/q, \Box q/r$]
12.	$(\Box(p\&q) \supset \Box q) \supset (\Box(p\&q) \supset (\Box p\&\Box q))$	(5), (11)×MP
13.	$\Box(p\&q) \supset (\Box p\&\Box q)$	(5), (11)×MP

You should go through the proof of *K2* yourself. Here's the proof of *K3*, which shows how earlier theorems can be appealed to in new proofs.

K3 $\Box(p\&q) \equiv \Box p\&\Box q$

1.	$\Box(p\&q) \supset (\Box p\&\Box q)$	K1
2.	$\Box(p\&q \supset p)$	(1)×N
3.	$\Box(p \supset q) \supset (\Box p \supset \Box q)$	K
4.	$\Box(p\&q \supset p) \supset (\Box p\&q \supset \Box p)$	(3),[p&q/p, p/q]
5.	$\Box p\&q \supset \Box p$	(2),(4)×MP
6.	$p\&q \supset q$	PC2
7.	$\Box(p\&q \supset q)$	(5)×N
8.	$\Box(p\&q \supset q) \supset (\Box p\&q \supset \Box q)$	(3),[p&q/p]
9.	$\Box p\&q \supset \Box q$	(7),(8)×MP
10.	$(p \supset q) \supset ((p \supset r) \supset (p \supset (q\&r)))$	PC3
11.	$(\Box(p\&q) \supset \Box p) \supset ((\Box(p\&q) \supset \Box q) \supset (\Box(p\&q) \supset (\Box p\&\Box q)))$	(10), $[\Box(p\&q)/p, \Box p/q, \Box q/r]$
12.	$(\Box(p\&q) \supset \Box q) \supset (\Box(p\&q) \supset (\Box p\&\Box q))$	(5), (11)×MP
13.	$\Box(p\&q) \supset (\Box p\&\Box q)$	(5), (11)×MP

Derived Rules

These are not strictly part of the system, but informal compression devices which we convince ourselves by any means necessary don't allow the proof of anything not provable already, before they were introduced. They're verboten unless each of their applications can be justified by regular rules albeit at much greater length.

DR2 $\vdash p\equiv q \rightarrow \vdash \Box p\equiv q$

1.	$\alpha \equiv \beta$	Given
2.	$\alpha \supset \beta$	(1)×PC
3.	$\Box\alpha \supset \Box\beta$	(2)×DR1
4.	$\beta \supset \alpha$	(1)×PC
5.	$\Box\beta \supset \Box\alpha$	(4)×DR1
6.	$\Box\alpha \supset \Box\beta$	(3), (5)×PC

Now a derived rule which is difficult even to state much less prove. It's called *Substitution of Equivalents*, *Eq* for short. What it does is extend DR2 to all sentential environments whatsoever.

Eq $\vdash \alpha\equiv\beta$ and $\vdash\varphi(\alpha) \rightarrow \vdash\varphi(\beta/\alpha)$.

If two wffs are provably equivalent, then you can put one for the other in any theorem you like and the result will still be a theorem. *Eq* is proved by mathematical induction. You show first that the result holds for simple φ's and then that if it holds for φ and ψ, it must hold too for $\neg\varphi$ and $\varphi\lor\psi$ and $\Box\varphi$. Part of why **DR2** was proved first is that it helps with that final step, closure under necessitation.

Back when (not *way* back when) we said that $\Box$ and $\Diamond$ are "dual" in the sense that (i) $\Box p$ is equivalent to $\neg\Diamond\neg p$ and (ii) $\Diamond p$ is equivalent to $\neg\Box\neg p$. Is this duality respected by System K? Certainly (ii) holds, for that is how we defined $\Diamond$. But what about (i)? That'll be our next theorem.

K5 $\Box p \equiv \neg\Diamond\neg p$

1.	$p \equiv \neg\neg p$	PC12 (DN)
2.	$\Box p \equiv \neg\neg\Box p$	(1) $[\Box p/p]$
3.	$\Box p \equiv \neg\neg\Box\neg\neg p$	(2)×(1)×Eq
4.	$\Box p \equiv \neg\Diamond\neg p$	Def$\Diamond$

Now we can replace $\Box$ by $\neg\Diamond\neg$ anywhere we like in a theorem, and vice versa; and likewise with $\Diamond$ and $\neg\Box\neg$ (that we had already by the definition of $\Diamond$). A generalization of this is as follows; it's called $\Box$-$\Diamond$ *Interchange*, or LMI:

If α is a theorem, and β is the result of (i) replacing any sequence of modal operators by the "negative" of that sequence (boxes go in for diamonds and vice versa), and (ii) inserting or deleting a single $\neg$ both before and after the sequence, then ? is a theorem too.

Proof Sketch: Let $A_1...A_n$ be a sequence of boxes and diamonds, that is., each A_k is either $\Box$ or $\Diamond$. Let A_k' be $\Box$ if A_k is $\Diamond$ and vice versa. We start by showing that

$\vdash A_1...A_n p \equiv \neg A_1'...A_n' \neg p$

By PC we have

$\vdash A_1...A_n p \equiv A_1...A_n p$

On the right hand side, replace each $\Box$ by $\neg\Diamond\neg$ and each $\Diamond$ by $\neg\Box\neg$. This can be done by K5 and the definition of $\Diamond$, using derived rule Eq. This yields

$\vdash A_1...A_n p \equiv \neg A_1' \neg\neg \,...\neg\neg A_n' \neg p$

Now use DN and Eq to eliminate all the double negations and the result is (*). That is still not quite LMI but it's on the way. To show that the two sides are intersubstitutable everywhere, and no matter what wff replaces p, use US and Eq. Next time: a few more theorems and then we move on to validity in and soundness of K.

24.244 Modal Logic
Spring 2015