Sentential Calculus Revisited: Boolean Algebra

Modern logic began with the work of George Boole, who treated logic algebraically. Writing "+," "·," and "-" in place of "or," "and," and "not," he developed the sentential calculus in direct analogy to the familiar theories of groups, rings, and fields. Now that we have the predicate calculus with identity, we can discern the consequences of the axioms Boole wrote down, and we'll find that Boole's algebraic formalism got it exactly right. Here are the principles he discovered:

> **Definition.** The *language of Boolean algebra* is the language whose individual constants are "0" and "1," whose function signs are the unary function sign "-" and the binary function signs "+" and "·," and whose predicates are the binary predicates "=" and "≤."

For readability, we write "(x+y)" and (x·y)" in place of "+(x,y)" and "·(x,y)." So the terms of the language of Boolean algebra will constitute the smallest class of expressions which contains the variables and "0" and "1," contains -τ whenever it contains τ , and contains (τ+ρ) whenever it contains τ and ρ. We have unique readability, as usual.

> **Definition.** A structure is a *Boolean algebra* just in case it satisfies the following axioms:
>
> *Associative laws*:
> $(\forall x)(\forall y)(\forall z)((x+y)+z) = (x+(y+z))$
> $(\forall x)(\forall y)(\forall z)((x\cdot y)\cdot z) = (x\cdot(y\cdot z))$
>
> *Commutative laws*:
> $(\forall x)(\forall y)(x+y) = (y+x)$
> $(\forall x)(\forall y)(x\cdot y) = (y\cdot x)$
>
> *Idempotence*:
> $(\forall x)(x+x) = x$
> $(\forall x)(x\cdot x) = x$
>
> *Distributive laws*:
> $(\forall x)(\forall y)(\forall z)(x + (y\cdot z)) = ((x+y)\cdot(x+z))$
> $(\forall x)(\forall y)(\forall z)(x\cdot(y+z)) = ((x\cdot y) + (x\cdot z))$
>
> *Identity elements*:
> $(\forall x)(x+0) = x$
> $(\forall x)(x\cdot 1) = x$
>
> *Complementation laws*:
> $(\forall x)(x+-x) = 1$
> $(\forall x)(x\cdot -x) = 0$

Non-triviality:
$\neg 0=1$
Definition of "≤":
$(\forall x)(\forall y)(x \leq y \leftrightarrow (x+y)=y$

Definition. A sentence that's true in every Boolean algebra is a *theorem of Boolean algebra.*

Among the theorems of Boolean algebra are the universal closures of the following formulas:

Alternative definition of "≤":
$x \leq y \leftrightarrow (x \cdot y) = x$

Further principles governing 1 and 0:
$(x + 1) = 1$
$(x \cdot 0) = 0$
$((\forall x)(x+y) = y \leftrightarrow y=1)$
$((\forall x)(x \cdot y = y \leftrightarrow y=0)$
$((\forall x)(x+y) = x \leftrightarrow y=0)$
$((\forall x)(x \cdot y = x \leftrightarrow y=1)$

Further principles governing "-":
$-1 = 0$
$-0 = 1$
$--x = x$
$(x = -y \leftrightarrow ((x+y) = 1 \wedge (x \cdot y) = 0))$
$x \leq y \leftrightarrow -y \leq -x$

De Morgan's laws:
$-(x+y) = (-x \cdot -y)$
$-(x \cdot y) = (-x + -y)$

Absorption principles:
$(x + (x \cdot y)) = x$
$(x \cdot (x+y)) = x$

≤ *is reflexive*:
$x \leq x$

≤ *is transitive*:
$((x \leq y \wedge y \leq z) \rightarrow x \leq z)$

$\leq$ is antisymmetric:
$((x \leq y \wedge y \leq x) \rightarrow x = y)$

Lattice principles:
$x \leq 1$
$0 \leq x$
$((\forall x)x \leq y \leftrightarrow y = 1)$
$((\forall x)y \leq x \leftrightarrow y = 0)$
$(x \leq (x+y) \wedge y \leq (x+y))$
$((x\cdot y) \leq x \wedge (x\cdot y) \leq y)$
$((x \leq z \wedge y \leq z) \leftrightarrow (x+y) \leq z)$
$((z \leq x \wedge z \leq y) \leftrightarrow z \leq (x\cdot y))$

Further characterizations of $\leq$:
$x \leq y \leftrightarrow (-x + y) = 1$
$x \leq y \leftrightarrow (x \cdot -y) = 0$
$(x \leq y \leftrightarrow (\exists z)(y\cdot z)=x)$
$(x \leq y \leftrightarrow (\exists z)(\exists w)(x+z) = (y\cdot w)$

Tests for equality:
$(x = y \leftrightarrow ((x + -y) = 1 \wedge (-x + y) = 1))$
$(x = y \leftrightarrow ((x \cdot -y) =0 \wedge (-x \cdot y) = 0))$
$(x = y \leftrightarrow (x+y) = (x\cdot y))$

The simplest example is the two-element Boolean algebra, whose elements are the integers 0 and 1. “0” denotes 0, and “1” denotes 1. The sum of two elements of the domain is defined to be their maximum, the product is their minimum, and the complement of x is defined to be 1 - x. In a table,

a	*b*	$(a+b)$	$(a\cdot b)$	$-a$
1	1	1	1	0
1	0	1	0	0
0	1	1	0	1
0	0	0	0	1

Definition. The *augmented language of Boolean algebra* is the language obtained from the language of Boolean algebra by adding infinitely many additional constants $c_0, c_1, c_2, c_3,$....

Definition. For τ a term of the augmented language of Boolean algebra, the *dual* of τ, τ^d, is the term obtained from τ by exchanging “0” and “1” everywhere and exchanging “+” and “$\cdot$” everywhere.

Duality Principle. Let τ and ρ be terms of the augmented language of Boolean algebra. If the universal closure of $\tau=\rho$ is a theorem of Boolean algebra, so is the universal closure of $\tau^d = \rho^d$. If the universal closure of $\tau\leq\rho$ is a theorem of Boolean algebra, so is $\rho^d \leq \tau^d$.

Proof: Suppose that $\mathcal{B}$ **is a Boolean algebra and** σ is a variable assignment for $\mathcal{B}$ **that fails to satisfy $\tau^d = \rho^d$ in $\mathcal{B}$. Let the interpretation $\mathcal{A}$ be defined as follows:**

$|\mathcal{A}| = |\mathcal{B}|$
$\mathcal{A}(c_i) = \mathcal{B}(c_i)$, for each i
$\mathcal{A}$("1") = $\mathcal{B}$("0")
$\mathcal{A}$("0") = $\mathcal{B}$("1")
x "+"$^{\mathcal{A}}$ y = x "·"$^{\mathcal{B}}$ y
x "·"$^{\mathcal{A}}$ y = x "+"$^{\mathcal{B}}$ **y**
"-"$^{\mathcal{A}}$ x = "-"$^{\mathcal{B}}$ x
$\mathcal{A}$("≤") = {<y,x>: <x,y> ∈ "≤"$^{\mathcal{B}}$

It is easy the verify that, for each term μ and variable assignment δ, $\text{Den}_{\delta,\mathcal{A}}(\mu) = \text{Den}_{\delta,\mathcal{B}}(\mu^d)$. Consequently, σ fails to satisfy $\tau=\rho$ in $\mathcal{A}$. It is also easy to verify, by examining the axioms one by one, that $\mathcal{A}$ is a Boolean algebra. So $\mathcal{A}$ is a Boolean algebra in which the universal closure of $\tau=\rho$ is false.

We prove the second part of the Duality Principle — the part about "≤" — by deriving it from the first part. Suppose that the universal closure of $\tau \leq \rho$ is a theorem of Boolean algebra. It follows by the definition of "≤" that the universal closure of $(\tau+\rho) = \rho$ is a theorem of Boolean algebra. Using the first part of the Duality Principle, we conclude that the universal closure of $(\tau+\rho)^d = \rho^d$, which is the same as $(\tau^d \cdot \rho^d) = \rho^d$, is a theorem of Boolean algebra. It follows from the alternative definition of "≤," together with the commutative law for "·," that $\rho^d \leq \tau^d$ is a theorem of Boolean algebra.X

We now make the connection between Boolean algebra and the sentential calculus. Let us fix our attention on some SC language whose atomic sentences have been arranged in an infinite list.

Definition. For φ an SC sentence whose only connectives are "∨," "∧," and "¬," let us take the *algebraic translation* of φ, φ^a, to be the closed term of the augmented language of Boolean algebra, obtained as follows:

Replace the ith atomic sentence, wherever it occurs, by c_i, and replace "$\vee$," "$\wedge$," and "$\neg$," wherever they occur, by "+," "·," and "-," respectively.
Definition. If Γ is a set of SC sentences whose only connectives are "$\vee$," "$\wedge$," and "$\neg$," $\Gamma^a = \{\gamma^a: a \in \Gamma\}$.

Theorem. Let φ and ψ be SC sentences whose only connectives are "$\vee$," "$\wedge$," and "$\neg$." We have the following:
(a) φ is a tautology iff $\varphi^a = 1$ is a theorem of Boolean algebra.
(b) φ is a contradiction iff $\varphi^a = 0$ is a theorem of Boolean algebra.
(c) φ and ψ are logically equivalent iff $\varphi^a = \psi^a$ is a theorem of Boolean algebra.
(d) φ implies ψ iff $\varphi^a \leq \psi^a$ is a theorem of Boolean algebra.

Proof: We present the proof in the following wacky order: First, we prove the right-to-left direction of (c), then the give the left-to-right direction of (b), and finally we fill in the rest.

(c) $\Leftarrow$ Suppose that φ and ψ are not logically equivalent. Define an interpretation $\mathcal{A}$ **of the augmented language of Boolean algebra, as follows: For χ an SC sentence, let $[\chi]$ be the set of all formulas that are logically equivalent to χ; $[\chi]$ is called the *logical equivalence class* of** χ.

$|\mathcal{A}| = \{[\chi]: \chi$ an SC formula$\}$.
$\mathcal{A}(c_i)$ = the logical equivalence class of the ith atomic sentence.
$1^{\mathcal{A}}$ = the set of tautologies.
$0^{\mathcal{A}}$ **= the set of inconsistent sentences.**
For a and b elements of $|\mathcal{A}|$, define a "+"$^{\mathcal{A}}$ b to be the unique member c of $|\mathcal{A}|$ such that there exist sentences χ and θ with $a = [\chi]$, $b = [\theta]$, and $c = [(\chi \vee \theta)]$.

In order for this definition to make sense, we have to persuade ourselves that, for any a and b in $|\mathcal{A}|$**, there is indeed one and only one element c of $|\mathcal{A}|$ that meets the condition; but that's easy.**

For a and b elements of $|\mathcal{A}|$, define a "·"$^{\mathcal{A}}$ b to be the unique member c of $|\mathcal{A}|$ such that there exist sentences χ and θ with $a = [\chi]$, $b = [\theta]$, and $c = [(\chi \wedge \theta)]$.

For a an element of $|\mathcal{A}|$, define "-"$^{\mathcal{A}}$ to be the unique member c of $|\mathcal{A}|$ such that there exists a sentence χ with $a = [\chi]$ and $c = [\neg\chi]$.

It is routine to verify that $\mathcal{A}$, so defined, is a Boolean algebra. For example, the commutative law holds for "+" because, for any sentences χ and θ, $(\chi \vee \theta)$ is logically equivalent to $(\theta \vee \chi)$. The associative law holds for "·" because, for any sentences χ, θ, and η, $((\chi \wedge \theta) \wedge \eta)$ is logically equivalent to $(\chi \wedge (\theta \wedge \eta))$. And so on.

It is also straightforward to verify that, for any formula χ whose only connectives are "$\vee$," "$\wedge$," and "$\neg$," $\mathcal{A}(\chi^a) = [\chi]$. Since $[\varphi] \neq [\psi]$, $\mathcal{A}$ **is a Boolean algebra in which** $\varphi^a = \psi^a$ **fails.**
(b) ⇒ We define a set Δ of closed terms to be *a-inconsistent* (for *algebraically inconsistent*) just in case there exist elements $\delta_1, \delta_2, \ldots, \delta_n$ of Δ such that the sentence $(\delta_1 \cdot (\delta_2 \cdot \ldots \cdot \delta_n) \ldots) = 0$ is a theorem of Boolean algebra. a-consistent sets will play a role analogous to the role played by d-consistent sets in the proof of the Completeness Theorem. We have the following:

Lemma. For any set of closed terms Ω and closed term τ, if $\Omega \cup \{\tau\}$ and $\Omega \cup \{-\tau\}$ are both a-inconsistent, then Ω is a-inconsistent.

Proof: Exactly analogous to the corresponding proof for the sentential calculus.X

If $\varphi^a = 0$ isn't a theorem of Boolean algebra, $\{\varphi^a\}$ is a-consistent. Listing the sentences whose only connectives are "$\vee$," "$\wedge$," and "$\neg$" as $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \ldots$, we define:

$\Gamma_0 = \{\varphi\}$
If $\Gamma_n{}^a \cup \{\zeta_n{}^a\}$ is a-consistent, then $\Gamma_{n+1} = \Gamma_n \cup \{\zeta_n\}$.
Otherwise, $\Gamma_{n+1} = \Gamma_n \cup \{\neg\zeta_n\}$.

It follows from the lemma that $\Gamma_{n+1}{}^a$ is a-consistent if $\Gamma_n{}^a$ is. Consequently, if we define

Γ_∞ = the union of the Γ_ns,

we see that $\Gamma_\infty{}^a$ is a-consistent. Moreover, for each sentence ζ, either $\zeta \in \Gamma_\infty$ or $\neg\zeta \in \Gamma_\infty$.

Define a N.T.A. $\Im$ by stipulating that

$\Im(\chi) = 1$ iff $\chi \in \Gamma_\infty$,

for χ atomic. We want to see that, for any SC sentence χ whose only connectives are "$\vee$," "$\wedge$," and "$\neg$," $\Im(\chi) = 1$ iff $\chi \in \Gamma_\infty$. Since $\varphi \in \Gamma_\infty$, this will tell us that $\Im$ is a N.T.A. under which φ is true, hence that φ isn't a contradiction.

We proceed by *reductio ad absurdum*, supposing that the statement we're trying to prove is false, and deriving a contradiction. Let χ be a simplest counterexample to the statement we're trying to prove; that is, χ is a simplest sentence whose only connectives are "$\vee$," "$\wedge$," and "$\neg$" such that it's not the case that $\Im(\chi) = 1$ iff $\chi \in \Gamma_\infty$. We want to show that this supposition leads us to a contradiction. There are four cases:

Case 1. χ atomic. This is impossible, since $\Im(\chi) = 1$ iff $\chi \in \Gamma_\infty$ holds by the definition of $\Im$.

Case 2. χ is a disjunction, say $(\mu \vee \nu)$. There are two subcases:

Subcase 2a. $\Im(\chi) = 1$ and $\chi \notin \Gamma_\infty$. So either $\Im(\mu) = 1$ or $\Im(\nu) = 1$. Since μ and ν are both simpler than χ, it follows that either $\mu \in \Gamma_\infty$ or $\nu \in \Gamma_\infty$, and so either μ^a or ν^a is in $\Gamma_\infty{}^a$. Since $(\mu \vee \nu)$ isn't in Γ_∞, $\neg(\mu \vee \nu)$ is in Γ_∞, and so $(\neg(\mu \vee \nu))^a = -(\mu^a + \nu^a)$ is in Γ_∞. But this is impossible. Since $\mu^a \cdot -(\mu^a + \nu^a)) = 0$ and $\nu^a \cdot -(\mu^a + \nu^a)) = 0$ are both theorems of Boolean algebra, no a-consistent set can contain both μ^a and $-(\mu^a + \nu^a)$, and no a-consistent set can contain both ν^a and $-(\mu^a + \nu^a)$.

Subcase 2b. $\Im(\chi) \neq 1$ and $\chi \in \Gamma_\infty$. So neither $\Im(\mu)$ nor $\Im(\nu)$ is equal to 1. So, since μ and ν are both simpler than χ, neither μ nor ν is in Γ_∞. So both $\neg\mu$ and $\neg\nu$ are in Γ_∞. But this means that $(\neg\mu)^a = -(\mu^a)$ and $-(\nu^a)$ are both in $\Gamma_\infty{}^a$. But this is impossible: No a-consistent set can contain both $-(\mu^a)$, $-(\nu^a)$, and $\chi^a = (\mu^a + \nu^a)$, because $(-(\mu^a)\cdot(-(\nu^a)\cdot(\mu^a + \nu^a))) = 0$ is a theorem of Boolean algebra.

Case 3. χ is a conjunction. Similar.

Case 4. χ is a negation. Similar.

(a)$\Rightarrow$ If φ is a tautology, then $\neg\varphi$ is a contradiction. By the left-to-right direction of (b), $(\neg\varphi)^a = 0$ is a theorem of Boolean algebra. Since $(\neg\varphi)^a = -(\varphi^a)$, $\varphi^a = 1$ is a theorem of Boolean algebra.

(a)$\Leftarrow$ Take an atomic sentence η. If $\varphi^a = 1$ is a theorem of Boolean algebra, $\varphi^a = (\eta \vee \neg\eta)^a$ is a theorem of Boolean algebra. So, by the right-to-left direction of (c), φ is logically equivalent to $(\eta \vee \neg\eta)$. Hence, φ is tautological.

(b) $\Leftarrow$ If $\varphi^a = 0$ is a theorem of Boolean algebra, $(\neg\varphi)^a = 1$ is a theorem of Boolean algebra. So by the right-to-left direction of (a), $\neg\varphi$ is a tautology. So φ is a contradiction.

(c) $\Rightarrow$ If φ and ψ are logically equivalent, then $(\varphi \vee \neg\psi)$ and $(\neg\varphi \vee \psi)$ are both tautologies. It follows by the left-to-right direction of (1) that $(\varphi \vee \neg\psi)^a = 1$ and $(\neg\varphi \vee \psi) = 1$ are both theorems of Boolean algebra. In other words, $(\varphi^a + -\psi^a) = 1$ and $(-\varphi^a + \psi^a) = 1$ are both theorems of Boolean algebra. It follows by one of our "tests for equality" that $\varphi^a = \psi^a$ is a theorem of Boolean algebra.

(d) φ implies ψ
iff $(\neg\varphi \vee \psi)$ is a tautology
iff $(\neg\varphi \vee \psi)^a$ is a theorem of Boolean algebra [by (1)]
iff $(-\varphi^a + \psi^a) = 1$ is a theorem of Boolean algebra
iff $\varphi^a \leq \psi^a$ is a theorem of Boolean algebra [by one of our "further characterizations of $\leq$"]X̲

> **Corollary.** There is an algorithm for determining whether the universal closure of an equation $\tau = \rho$ of the augmented language is a theorem of Boolean algebra.

Proof: We can simplify the proof by making use of the following general fact, which follows immediately from the observation that Universal Generalization is a valid rule of proof:

> **Fact.** For Γ a set of sentences and χ a formula, let χ^c be a sentence obtained from χ by replacing all its free variables by new constants — that is, constants that do not appear either in χ or the members of Γ — in such a way that different occurrences of the same variable are replaced by the same constant and occurrences of different variables are replaced by different constants. Then the universal closure of χ is a logical consequence of Γ if and only if χ^c is a logical consequence of Γ.

Because of the fact, we can restrict our attention to the case in which τ and ρ are closed. We can find sentences φ and ψ such that $\varphi^a = \tau$ and $\psi^a = \rho$ are both theorems of Boolean algebra by performing the following operations: First, replace each occurrence of c_i by the ith atomic sentence. Second, replace each occurrence of "1" by some tautology and each occurrence of "0" by some contradiction. Finally, replace each occurrence of "+," "·," and "-" by "$\vee$," "$\wedge$," and "$\neg$," respectively.

By (c), to test whether $\tau = \rho$ is a theorem of Boolean algebra, it will suffice to determine whether φ and ψ are logically equivalent, which we can do either by truth tables or by the search-for-counterexample method.X

Sentential logic is one of the places where Boolean algebra is useful, but it's not the only place. Another application is to the theory of sets. In order to exhibit the connection between sentential logic, Boolean algebra, and set theory, we make the following:

> **Definition.** A *power set interpretation* of the augmented language of Boolean algebra is an interpretation $\mathcal{A}$ **that meets the following conditions:**
> **For some nonempty set S, $|\mathcal{A}|$ is the set of all subsets of S. ($|\mathcal{A}|$ is what set theorists call the *power set* of S.)**
> **"1"$^{\mathcal{A}} = S$.**
> **"0"$^{\mathcal{A}} = \varnothing$, the empty set.**
> **For any subsets A and B of S, A "+"$^{\mathcal{A}}$ $B = A \cup B$, which is, by definition, $\{x\colon x \in A$ or $x \in B\}$.**
> **A "·"$^{\mathcal{A}}$ $B = A \cap B$, which is, by definition, $\{x\colon x \in A$ and $x \in B\}$.**
> **"-"$^{\mathcal{A}}$ $A = S \sim A$, which is, by definition, $\{x\colon x \in S$ and $x \notin A\}$.**

Notice the correlation between the set theoretic operations and the logical operations of sentential calculus: For any subsets A and B of S and any element c of S, we have:

> c is an element of $A \cup B$ iff c is an element of A or c is an element of B.
> c is an element of $A \cup B$ iff c is an element of A and c is an element of B.
> c is an element of $-A$ iff c is not an element of A.

This is the connection we wish to exploit.

> Theorem. For any terms τ and ρ of the augmented language, the universal closure of $\tau = \rho$ is a theorem of Boolean algebra if and only if it's true in every power set interpretation of the language.

Proof: The left-to-right direction is just a matter of checking that every power set interpretation is a Boolean algebra. Nothing to it.

For the right-to-left direction, we begin by noting, just as in the proof of the last corollary, that we may restrict our attention to the case in which τ and ρ are closed. Assuming that $\tau = \rho$ is not a theorem of Boolean algebra, find SC sentences φ and ψ such that $\varphi^a = \tau$ and $\psi^a = \rho$ are theorems of Boolean algebra. Since every theorem of Boolean algebra is true in every power set interpretation, in order to find a power set interpretation in which $\tau = \rho$ is false, it will suffice to find a power set interpretation under which $\varphi^a = \psi^a$ is false.

Let $\mathcal{A}$ be a power set interpretation, defined as follows:

> $|\mathcal{A}|$ = the power set of the set of all N.T.A.s.
> "1"$^{\mathcal{A}}$ = the set of all N.T.A.s.
> "0"$^{\mathcal{A}}$ = the empty set.
> $c_i^{\mathcal{A}}$ = the set of all N.T.A.s under which the ith atomic sentence is true.

We want to see that, for any sentence χ whose only connectives are "$\vee$," "$\wedge$," and "$\neg$," $\chi^{a\mathcal{A}}$ = the set of N.T.A.s under which χ is true. Let Σ be the set of all sentences χ, whose only connectives are "$\vee$," "$\wedge$," and "$\neg$," such that $\chi^{a\mathcal{A}}$ = the set of N.T.A.s under which χ is true. We need to see that Σ contains the atomic sentences and that it contains $(\chi \vee \theta)$, $(\chi \wedge \theta)$, and $\neg\chi$, whenever it contains χ and θ.

Σ contains all atomic sentences. If χ is the ith atomic sentence, $\chi^{a\mathcal{A}} = c_i^{\mathcal{A}}$ = the set of N.T.A.s under which χ is true.

Σ is closed under "$\vee$." Assume that χ and θ are in Σ. $(\chi \vee \theta)^{a\mathcal{A}} = (\chi^a + \theta^a)^{\mathcal{A}} = \chi^{a\mathcal{A}} \cup \theta^{a\mathcal{A}}$ = {N.T.A.s under which χ is true} $\cup$ {N.T.A.s under which θ is true} = {N.T.A.s under which either χ or θ is true} = {N.T.A.s under which $(\chi \vee \theta)$ is true}.

Σ is closed under "$\wedge$" and "$\neg$." Similar.

Since φ and ψ are not logically equivalent, {N.T.A.s under which φ is true} is not equal to {N.T.A.s under which ψ is true}. Hence $\varphi^{a_{\mathcal{A}}} \neq \psi^{a_{\mathcal{A}}}$, so $\mathcal{A}$ is a power set interpretation in which $\varphi^{a} = \psi^{a}$ is false, and so a power set interpretation in which $\tau=\rho$ is false.$\underline{\mathrm{X}}$