24.118: Paradox and Infinity, Spring 2019
Problem Set 3: Omega-Sequence Paradoxes

How these problems will be graded:

- In Part I there is no need to justify your answers. Assessment will be based on whether your answers are correct.
- In Part II you must justify your answers. Assessment will be based both on whether you give the correct answer and on how your answers are justified. (In some problem sets I will ask you to answer questions that don't have clear answers. In those cases, assessment will be based entirely on the basis of how your answer is justified. Even if it is unclear whether your answer is correct, it should be clear whether or not the reasons you have given in support of your answer are good ones.)
- *No answer may consist of more than 150 words.* Longer answers will not be given credit. (Showing your work in a calculation, a proof, or a computer program does not count towards the word limit.)
- You may consult published literature and the web. You must, however, credit all sources. Failure to do so constitutes plagiarism and can have serious consequences. For advice about when and how to credit sources see: https://integrity.mit.edu/ (You do not need to credit course materials.)

> *Warning:* You may find this problem set harder than the previous two. Whereas problem Sets 1 and 2 were largely a test of your math skills, Problem Set 3 will begin to test your philosophical abilities. In working through the problems, you may find that the rules of the game are not as clear as you'd like—and therefore that you can't just rely on your math skills. Keep in mind that even if a problem is murky, it can be addressed in more or less thoughtful ways: a good answer will reveal that you've understood the complexity of the underlying terrain and that you've given the issue serious thought. Addressing a murky question requires that you be *more* careful, not less.

Part I

1. Imagine an island on which everyone is either a knight or a knave. Knights only assert truths, knaves only assert falsehoods.

   For each of the scenarios below, determine whether one can settle the question of whether S_0 a knight or a knave. If the answer is "yes", state whether S_0 is a knight or a knave; if the answer is "no", state the reason why.

(a) Ten islanders, $S_0, S_1, \ldots, S_9$, are lined up. S_0 is at the back of the line; in front of her is S_1; in front of him is S_2; and so on. Each islander says: "There is at least one person in front of me, and everyone in front of me is a knave." (5 points)

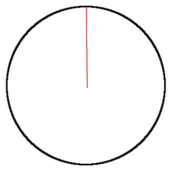
(b) Countably many islanders, $S_0, S_1, S_2, \ldots$, are lined up, one for each natural number. S_0 is still at the back of the line; in front of her is S_1; in front of him is S_2, and so on. Each islander says: "There is at least one person in front of me, and everyone in front of me is a knave." (5 points)

(c) As in the previous case, but this time each islander says: "There is at least one person in front of me, and everyone in front of me is a knight." (5 points)

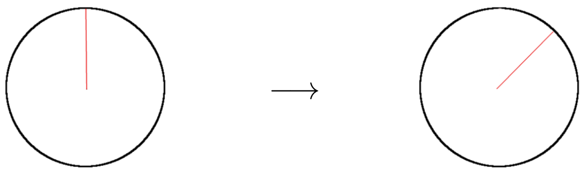
Part II

2. The following questions are intended to test your understanding of continuity assumptions.[1] They may or may not have determinate answers. (If you think a question has a determinate answer, make sure you explain why. If you think it doesn't, make sure you explain why not.)

   (a) You have a wheel with a radius of 1 meter. You draw a red line going from the center of the wheel to the twelve o'clock position:



   You then rotate the wheel, in steps. At each step, you rotate the wheel one radian clockwise, so that the outermost point of the red line travels 1 meter around the perimeter of the wheel. Here is the first rotation:



   Because the tip of the red line travels 1 meter around the perimeter after each rotation, and because the perimeter of the wheel is 2π meters (which is an

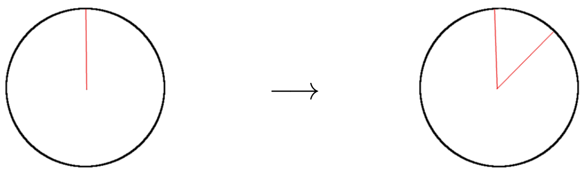
[1]I am told that the underlying puzzle was created by Oxford philosopher Frank Arntzenius's daughter, when she was very young.

irrational number of meters), there is no positive integer n such that, after n steps, the red line returns to a position it had occupied before.

Suppose you perform this operation infinitely many times, once for each positive integer. At noon the red line is at the twelve o'clock position. At 12:30 you rotate the wheel one radian clockwise. At 12:45 you rotate the wheel another radian clockwise. And so forth: for each $n \geq 1$, you perform the nth rotation $\frac{1}{2^n}$ hours before 1pm.

Question: in which direction will the red line be pointing at 1pm? (10 points)

(b) Consider a variant of the case in part (a). As before, you start by drawing a red line from the center of the wheel to the twelve o'clock position. But this time you draw new lines, in steps. At each step, you draw a red line at a one radian angle from the last line you drew. At noon you draw the initial red line, pointing to the twelve o'clock position. At 12:30 you draw an additional line, a radian away in the clockwise direction. At 12:45 you draw an additional line, a radian away in the clockwise direction. And so forth: for each $n \geq 1$, you draw a line $\frac{1}{2^n}$ hours before 1pm. Here are the first two steps of the process:



There is a well-defined fact of the matter about what the wheel will look like from noon onwards. Namely: a radial line r will be colored red if and only if, for some number n, r is exactly n radians away from twelve o'clock, going clockwise.

Now it's 1pm and you're done drawing lines on the wheel. The next step is to rotate the wheel.

Consider the result of rotating the wheel one radian in the clockwise direction. It'll look exactly the way the unrotated wheel would have looked if you had skipped drawing the 12pm line. And if you rotate the wheel one radian further, for a total rotation of two radians, it'll look exactly the way the unrotated wheel would have looked if you had skipped drawing the 12pm and 12:30pm lines. And so forth: after a total of n rotations, the wheel will look exactly the way the original unrotated wheel would have looked if you had skipped drawing all lines up to and including the 1pm$-\frac{1}{2^{n-1}}$ hours line.

Suppose that you rotate the wheel infinitely many times, once for each positive integer. At 1pm you rotate the wheel one radian in the clockwise direction. At 1:30 you rotate the wheel another radian in the clockwise direction. Again at 1:45. And so forth. For each $n \geq 0$, you perform the nth rotation $\frac{1}{2^n}$ hours before 2:00pm.

Question: What does the wheel look like at 2pm? (10 points)

(c) Suppose you start by drawing lines on the wheel as in part (b). But this time you perform no rotations. Instead, you *erase* red lines using the following procedure. At 1:00pm you erase the red line pointing to the twelve o'clock position. At 1:30 you erase the line pointing one radian in the clockwise direction from the last line erased. And so forth: for each $n \geq 1$, you erase the line one radian away in the clockwise direction from the last line erased.

Question: Is there a difference between the way our wheel looks at 2pm and the way the wheel in the previous exercise looked at 2pm? (5 points; make sure you explain your answer)

3. The following questions are also intended to test your understanding of continuity assumptions. As before, they may or may not have determinate answers. (If you think a question has a determinate answer, make sure you explain why. If you think it doesn't, make sure you explain why not.)

   (a) Fool has infinitely many dollar bills and has labelled each of them with a different natural number (its 'serial number'). One minute before midnight, Fool gives you a dollar bill. Half a minute later, he gives you two dollar bills. Fifteen seconds later, he gives you four dollar bills. And so forth. (For each $i \geq 0$, Fool gives you 2^i dollar bills 2^{-i} minutes before midnight.) There is, however, a catch. Each time you receive money from Fool, you are required to put together all your dollar bills, and burn the one with the lowest serial number. Assume that, at midnight, you have every dollar bill that you received from Fool and did not burn. How much money will you have at midnight? (10 points; make sure you explain your answer)

   (b) One minute before midnight, Fool gives you a dollar bill. Half a minute later, you burn a dollar bill. Fifteen seconds later, Fool gives you two dollar bills. 7.5 seconds later, you burn a dollar bill. 3.75 seconds later Fool gives you four dollar bills. And so forth. (For each $i \geq 0$, Fool gives you 2^i dollar bills 2^{-2i} minutes before midnight and you burn a dollar bill $2^{-(2i+1)}$ minutes before midnight.) Assume that, at midnight, you have every dollar bill that you received from Fool and did not burn. How much money will you have at midnight? (10 points; make sure you explain your answer)

4. Team A consists of an ω-sequence of persons, $a_0, a_1, a_2, \ldots$.[2] Between 11:00am and noon, they are each given an opportunity to choose a one or a zero. They proceed in inverse order, with a_0 choosing last, a_1 next to last, and so forth. To fix ideas, let us assume that choices made on the following schedule:

[2]This puzzle and the next are due to MIT philosopher Steve Yablo (of Yablo's Paradox fame).

time	chooser
$\vdots$	$\vdots$
12:00 $-(1-\frac{1}{2^k})$hours	a_k
$\vdots$	$\vdots$
11:15	a_2
11:30	a_1
12:00	a_0

$a_0, a_1, a_2, \ldots$ all want to pick the same color, so they agree on the following strategy:

Team A's Strategy If a_{k+1} chooses 0, a_k chooses 0; otherwise, a_k chooses 1.

Assume that every member of the team succeeds in following this strategy. As a result, they either all choose 1 or all choose 0. Is there a *causal explanation*[3] to be given about why the group chooses one rather than zero (or zero rather than one)? If so, spell out the explanation in some detail. If not, explain why not. (10 points)

5. There are two teams, Team A and Team B. Each team consists of an ω-sequence of persons: $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$, respectively. Between 11:00am and noon, the members of Team A and Team B will take turns choosing ones and zeroes. They proceed in inverse order, with a_0 and b_0 choosing last, a_1 and b_1 choosing next to last, and so forth. In addition, a_k always chooses ahead of b_k. To fix ideas, let us assume that choices made on the following schedule:

time	chooses first	chooses second
$\vdots$	$\vdots$	$\vdots$
12:00 $-(1-\frac{1}{2^k})$hours	a_k	b_k
$\vdots$	$\vdots$	$\vdots$
11:15	a_2	b_2
11:30	a_1	b_1
12:00	a_0	b_0

The two teams have opposing goals: Team A wants a_k and b_k to choose the same, for each $k \in \mathbb{N}$. Team B wants them to choose differently,

Since Team B has the advantage of choosing second, they agree on the following strategy:

Team B's Strategy If a_k chooses 0, b_k chooses 1; otherwise, b_k chooses 0.

[3]What is a causal explanation? To causally explain why x occurred rather than y is to describe a sequence of causes and effects leading to x in a way that makes clear why x came about rather than y.

Team A knows about Bacon's Puzzle. First, they partition the set of ω-sequences of 0s and 1s by placing two sequences in the same cell if and only if the sequences differ at most finitely (i.e. there are at most finitely many $i \in \mathbb{N}$ such that the sequences differ on the ith position). Then they agree on a representative from each cell.

Now suppose it is a_k's turn to choose and let B be the sequence $\langle c(b_0), c(b_1), c(b_2), \ldots \rangle$, where $c(b_i)$ is the choice that b_i has in fact made (or will in fact make). Since $b_0, \ldots, b_k$ have not made their choices yet, a_k doesn't know what they are. But we will suppose that a_k knows what $b_{k+1}, b_{k+2}, \ldots$'s choices are. Accordingly, a_k knows what lies in all but finitely many positions of the sequence B. This is enough to ascertain which cell B is in, and therefore to identify ρ_B, which is the sequence that had been previously been agreed upon as a representative for B's cell. a_k is then asked to choose as follows:

Team A's Strategy If the sequence ρ_B has a 0 in its kth position, a_k chooses 0; otherwise, a_k chooses 1.

(a) By assuming that every member of Team A succeeds in following Team A's Strategy, show that there are at most finitely many $k \in \mathbb{N}$ such that a_k and b_k choose differently. (5 points)

(b) It is obvious that if every member of Team B succeeds in following Team B's Strategy, a_k and b_k will choose differently for each $k \in \mathbb{N}$. So it follows form part (a) that it is impossible for every member of Team A to succeed in following Team A's strategy and every member of Team B to succeed in following Team B's strategy.

Does the setup of the problem settle the question of who will fail to follow their assigned strategy? Does it determine a causal mechanism that might explain why people who don't follow their assigned strategy fail to do so? (10 points)

6. There are two prisoners in a room. They both close their eyes and each of them is approached by a guard. Each guard flips a fair coin. If the coin lands Heads, she gives her prisoner a red hat; if it lands Tails, she gives her prisoner a blue hat. Once the prisoners have been assigned hats, they are both allowed to open their eyes.

As soon as they each see the color of the other's hat (but not the color of their own hat), the prisoners are taken into separate cells so that they are unable to communicate with one other. At that point, they are each to be asked to name the color of their hat. The guards will then proceed as follows:

- if at least one of the prisoners answers correctly, they will both be set free;
- otherwise, they will both be shot.

(a) Find a strategy that the prisoners can agree upon ahead of time which guarantees that their chance of survival is above 50%. Make sure your strategy is *deterministic*: it must determine a definite outcome for each prisoner, given the prisoner's situation at the time of his decision. (5 points)

(b) Given that the prisoners have no access to information about the color of their own hats, and that the colors were chosen using independent coin tosses, what explains the possibility of a deterministic strategy that brings the prisoners' chance of survival above 50%? (10 points)