Fuzzy and Rough Sets
Part I

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Aim

- Present aspects of fuzzy and rough sets.
- Enable you to start reading technical literature in the field of AI, particularly in the field of fuzzy and rough sets.
- Necessitates exposure to some formal concepts.

Overview Part I

- Types of uncertainty
- Sets, relations, functions, propositional logic, propositions over sets

= Basis for propositional rule based systems

Overview Part II

- Fuzzy sets
- Fuzzy logic
- Rough sets
- A method for mining rough/fuzzy rules
- Uncertainty revisited

Uncertainty

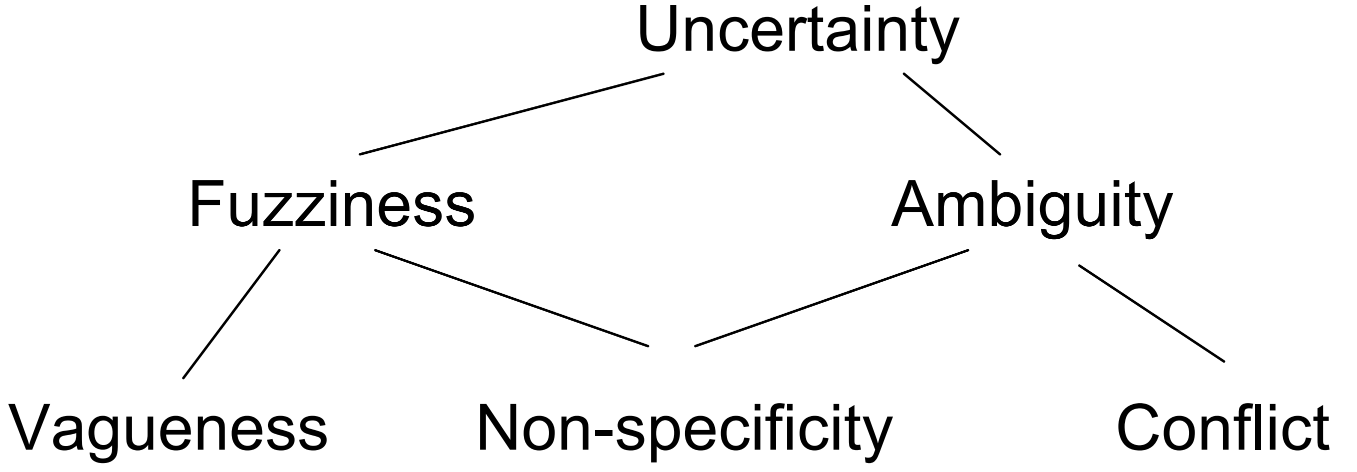
- What is uncertainty?
  – The state of being uncertain. (Webster).
- What does uncertain mean?
  – Not certain to occur.
  – Not reliable.
  – Not known beyond doubt.
  – Not clearly identified or defined.
  – Not constant. (Webster).

Uncertainty

- Ambiguity: existence of one-to-many relations
  - Conflict: distinguishable alternatives
  - Non-specificity: indistinguishable alternatives
- Fuzziness:
  - Lack of distinction between a set and it's complement (Yager 1979)
  - Vagueness: nonspecific knowledge about lack of distinction

Uncertainty

- Klir/Yuan/Rocha:



Model

- What is a model?
  - A mathematical representation (idealization) of some process

    (Smets 1994)
- Model of uncertainty:
  - A mathematical representation of uncertainty

Sets: Definition

- A *set* is a collection of *elements*
  - If *i* is a member of a set *S*, we write *i Î S,* if not we write *i Ï S*
  - *S = {1,2,3,4} = {4,1,3,2}* – explicit list
  - *S = {i Î* ***Z*** | *1£ i £ 4}* – defining condition
  - Usually: Uppercase letters denote sets, lowercase letters denote elements in sets, and functions.

Sets: Operations

- $A = \{1,2\}$, $B = \{2,3\}$ – sets of elements

  union:

  $A \cup B = \{1,2,3\} = \{i \mid i \in A$ or $i \in B\}$

  Intersection:

  $A \cap B = \{2\} = \{i \mid i \in A$ and $i \in B\}$

  Difference:

  $A - B = \{1\} = \{i \mid i \in A$ but not $i \in B\}$

Sets: Subsets

- A set *B* is a *subset* of *A* if and only if all elements in *B* are also in *A.* This is denoted

  B Í A.
- *{1,2} Í {2,1,4}*

Sets: Subsets

- The *empty set* ∅, containing nothing, is a subset of *all* sets.
- Also, note that *A* Í *A* for any *A*.

Sets: Cardinality

- For sets with a finite number of elements, the cardinality of a set is synonymous with the number of elements in the set.
- $|\{1,2,3\}| = 3$
- $|\varnothing| = 0$

Cartesian Product: Set of Tuples

- *(a,b)* is called an *ordered pair* or *tuple*
- The *cartesian product A ´ B* of sets *A* and *B,* is the set of all *ordered pairs* where the first element comes from *A* and the second comes from *B.*

  A ´ B = {(a,b) | a Î A and *b Î B}*
- *{1,2} ´ {3,2}* = *{(1,3),(1,2),(2,3),(2,2)}*

Relations: Subsets of Cartesian Products

- A relation *R* from *A* to *B* is a subset of $A \times B$
- $R \subseteq A \times B$
- $\{(1,2),(2,3)\}$ is a relation from $\{1,2\}$ to $\{3,2\}$
- $\{(1,3)\}$ is also a relation from $\{1,2\}$ to $\{3,2\}$

Binary Relations

- A relation from a set *A* to itself is called a *binary relation*, i.e., *R* Í *A* ´ *A* is a binary relation.
- Properties of a binary relation *R*:
  - *(a,a)* Î *R for all a* Î *A,*
    - *R* is **reflexive**
  - *(a,b)* Î *R* implies *(b,a)* Î *R,*
    - *R* is **symmetric**
  - *(a,b),(b,c)* Î *R* implies *(a,c)* Î *R,*
    - *R* is **transitive**

Relations: Equivalence and Partitions

- A binary relation on *A* is an *equivalence relation* if it is reflexive, symmetric and transitive.
- Let $R(a) = \{b \mid (a,b) \in R\}$

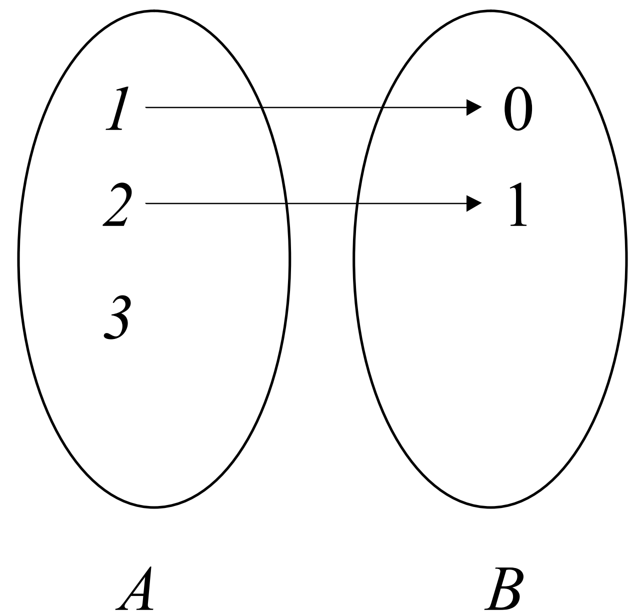
Relations: Equivalence and Partitions

- If *R* is an equivalence relation, then for *a,b* Î *R,* either
  - *R(a)* = *R(b)* or
  - *R(a)* Ç *R(b)* =Æ.
  - *R(a)* is called the *equivalence class of a under R*
- The different equivalence classes under *R* of the elements of A form what is called a *partition* of A

Functions: Single Valued Relations

- *R* Í *{a,b,c}* ´ *{1,2}*
- *R(*a*)* = *{1}*
- *R(*b*)* = *{2}*
- *R(*c*)* = ∅
- *|R(x)|£ 1 for all x, R* is *single valued*

- Is *R'* on the right single valued?



R' = *{(1,0),(2,1)}*

Functions: Partial and Total

- A single valued relation is called a *partial function*.
- A partial function *f* from *A* to *B* is *total* if $|f(a)| = 1$ for all *a* Î *A.* It is then said to be defined for all elements of *A.* Usually a total function is just called a function.

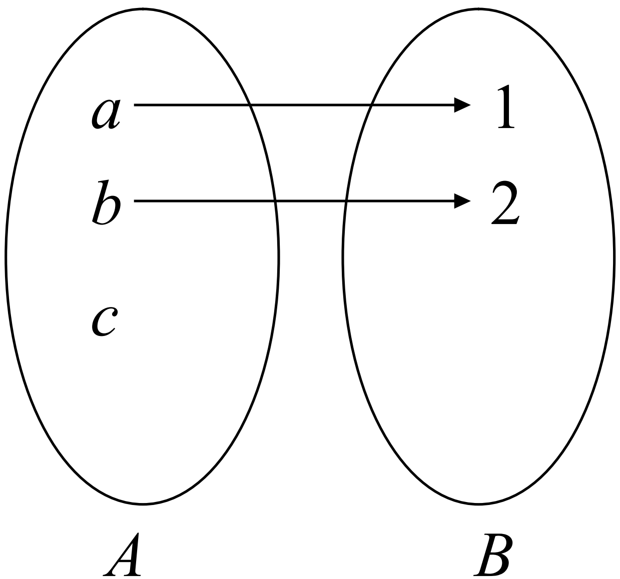
Functions: Partial and Total

- If a function *f* is from *A* to *B, A* is called the *domain* of *f,* while *B* is called the *co-domain* of *f.*
- A function *f* with domain *A* and co-domain *B* is often written

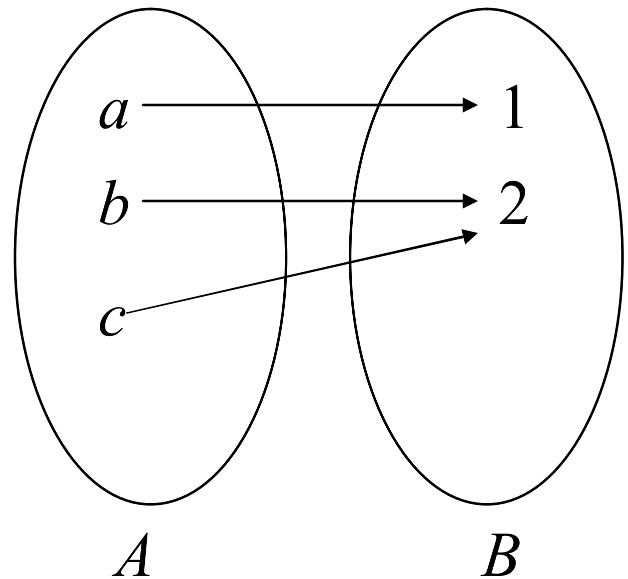
f:A ® *B.*

Functions: Extensions

- A partial function *g* such that *f* Í *g* is called an *extension* of *f.*



$f = \{(a,1),(b,2)\}$



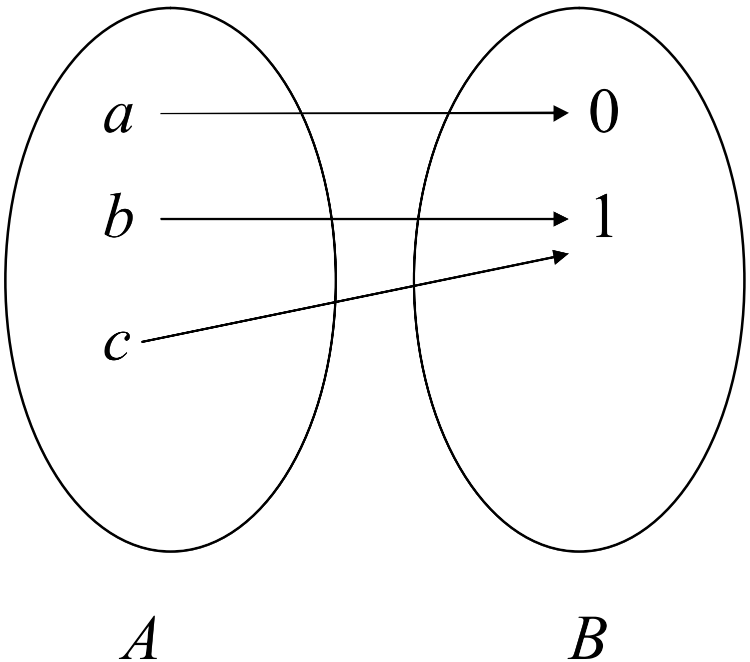
$g = \{(a,1),(b,2),(c,2)\}$

Characteristic functions: Sets

- A function *f:A ® {0,1}* is called a *characteristic function* of

  S = {a Î A | f(a) = 1}.
- *S Í A*

Characteristic functions: Sets



$f = \{(a,0),(b,1),(c,1)\}$

$S = ?$

Propositional Logic

- Proposition: statement that is either *true* or *false.*

- *"This statement is false." (Eubulides)*
- *If pain and ST-elevation, then MI.*

Patient is in pain and has ST-elevation.
What can we say about the patient?

Propositional language

- Language:
  - An infinite set of *variables*

    V={a, b, ...}
  - A set of *symbols {~, v, (,)}*
- Any string of elements from the above two sets is an *expression*
- An expression is a *legal* (well formed) formula (wff) or it is not

Propositional Syntax

- wff formation rules:
  - A variable alone is a wff
  - If α is a wff, so is $\sim\alpha$
  - If α and β are wff, so is $(\alpha \vee \beta)$

- Is (a v ~~~b) a wff?
- Is a v b a wff?

Propositional Operators

Truth functional

- Negation (not): ~

  ~α

	~
0	1
1	0

- Disjunction (or): v

  (α v β)

v	0	1
0	0	1
1	1	1

Semantics

- A *setting* s:V ® {0,1} assigning each variable either 0 or 1, denoting true or false respectively
- An Interpretation I_V:wff ® {0,1} used to compute the truth value of a wff

Semantics

- Variables

  $I(a) = s(a)$

- Composite wff:

  $I(\sim\alpha) = \sim I(\alpha)$

  $I(\alpha \vee \beta) = I(\alpha) \vee I(\beta)$

Semantics Example

$$I(\sim(\sim a \vee \sim b)) = \sim I(\sim a \vee \sim b)$$
$$= \sim(\sim I(a) \vee \sim I(b))$$
$$= \sim(\sim s(a) \vee \sim s(b))$$

If we let $s(a) = 1$, $s(b) = 0$

$$I(\sim(\sim a \vee \sim b)) = \sim(\sim 1 \vee \sim 0)$$
$$= \sim(0 \vee 1) = \sim 1 = 0$$

New Operator: And

- Conjunction (and): ^

  $(\alpha \wedge \beta) = \sim(\sim\alpha \vee \sim\beta)$

^	0	1
0	0	0
1	0	1

New Operator: Implication

- Implication (if...then): ®

  $(\alpha \text{ ® } \beta) = (\sim\alpha \text{ v } \beta)$

®	0	1
0	1	1
1	0	1

New Operator: Equivalence

- Equivalence: $\leftrightarrow$

  $(\alpha \leftrightarrow \beta) = (\alpha \text{ ® } \beta) \text{ ^ } (\beta \text{ ® } \alpha)$

$\leftrightarrow$	0	1
0	1	0
1	0	1

Semantics Of New Operators

- Conjunction:

  $I(\alpha \wedge \beta) = I(\alpha) \wedge I(\beta)$

- Implication:

  $I(\alpha \circledR \beta) = \sim I(\alpha) \vee I(\beta)$

- Equivalence:

  $I(\alpha \leftrightarrow \beta) = I(\alpha \circledR \beta) \wedge I(\beta \circledR \alpha)$

Propositional Consequence: A Teaser

- s = “Alf studies”
- g = “Alf gets good grades”
- t = “Alf has a good time”
  - (s ® g)
  - (~s ® t)
  - (~g ® ~t)

(~s v g) ^ (s v t) ^ (g v ~t) = g ^ (s v t)

At least Alf gets good grades.

Propositions Over a Set

- Propositions that describe properties of elements in a set
- Modeled by characteristic functions
- Example: *even*: N ® *{0,1}*

even(x) = (x+1) modulo 2

even(2) = 1

even(3) = 0

Truth Sets

- Truth set of proposition over U

  p:U ® *{0,1}*

  $T_U(p) = \{x \mid p(x) = 1\}$

- Example $T_N(\text{even}) = \{2,4,6,...\}$

Semantics

- Semantics are based on truth sets
  - $I_U(p(x)) = 1$ if and only if x in $T_U(p)$

- Following previous definitions, we have that
  - $T_U(\sim p) = U - T_U(p)$
  - $T_U(p \vee q) = T_U(p) \cup T_U(q)$
  - $T_U(p \wedge q) = T_U(p) \cap T_U(q)$

Semantics Example

- Two propositions over natural numbers
  - even
  - prime

$$T_N(\text{even} \wedge \text{prime}) = T_N(\text{even}) \cap T_N(\text{prime})$$

$$= \{2\}$$

$$I_N(\text{even}(x) \wedge \text{prime}(x)) = 1 \text{ if and only if } x=2$$

Inference: Modus Ponens

- Modus Ponens (rule of detachment):

α

α ® β

β

Ted is cold

If Ted is cold, he shivers

Ted shivers

An “implication-type rule application” mechanism

Next Time

- How to include uncertainty about set membership
- Extend this to logic
- A method for mining propositional rules