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HST.582J / 6.555J / 16.456J Biomedical Signal and Image Processing
Spring 2007

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HST-582J/6.555J/16.456J Biomedical Signal and Image Processing Spring 2007

Problem Set Y

- QUIZ 2 will take place on Thursday, May 10, from 9:30-11 am in 56-154 (usual time and place).
- The quiz will be closed book. Two $8\frac{1}{2} \times 11$ inch sheet of notes (both sides) will be allowed.
- The quiz will cover material presented in lectures from March 13 (PDFs) thru April 26 (Random Signals II). Note that there is some overlap with the image processing topics covered on Quiz 1.
- Coverage of topics on the quiz will be somewhat representative of the amount of time spent on each topic in lectures, labs and problem sets. You are *not* responsible for material in the course notes that was not covered elsewhere in the course. Please see the annotated outline for listings of specific topics.
- This ungraded problem set, which includes questions from with Quiz 2 from 2005 and 2006, will help you prepare for the quiz in two ways: First, it gives you a chance to work with recently presented material *that was not covered by previous problem sets*. Second, it illustrates the type of questions asked on quizzes in prior years.
- This problem set will not be collected or graded. Solutions will be posted on the course website on Monday, May 7.

Question 1

$x[n]$ is a stationary random signal. Each sample value of $x[n]$ is independent and uniformly distributed between $+1$ and -1 . $y[n]$ is the result of taking the first difference of $x[n]$, that is,

$$y[n] = x[n] - \frac{1}{2}x[n-1].$$

(a) Compute and sketch the autocorrelation functions $R_x[k]$ and $R_y[k]$.

(b)

- $y[n]$ is Gaussian true / false (circle one)

- $y[n]$ is stationary true / false (circle one)

- $y[n]$ is zero-mean true / false (circle one)

- $S_y(0) = 2$ true / false (circle one)
($S_y(f)$ denotes the power spectrum of $y[n]$.)

Question 2



(A)



(B)



(C)

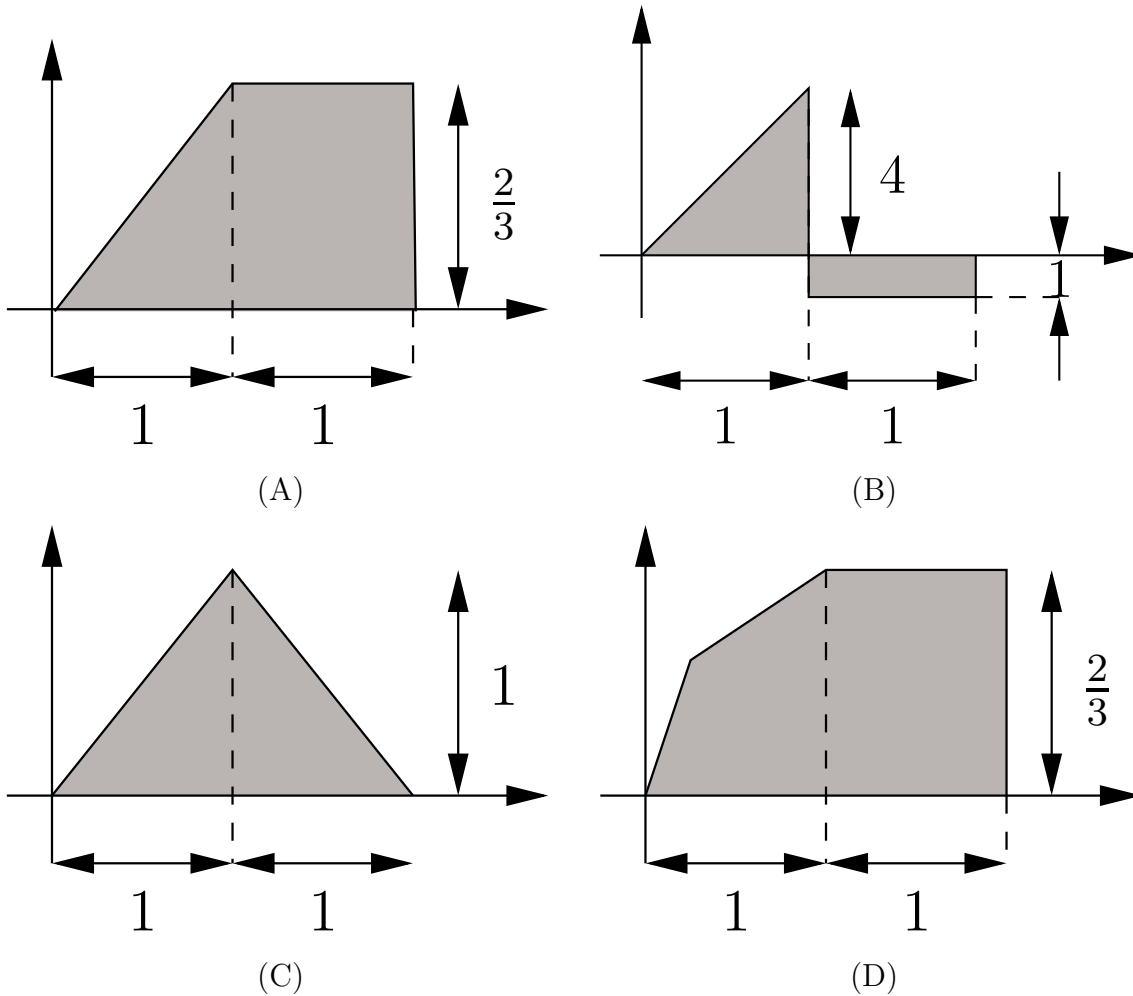


(D)

Identify the figure that corresponds to each description below:

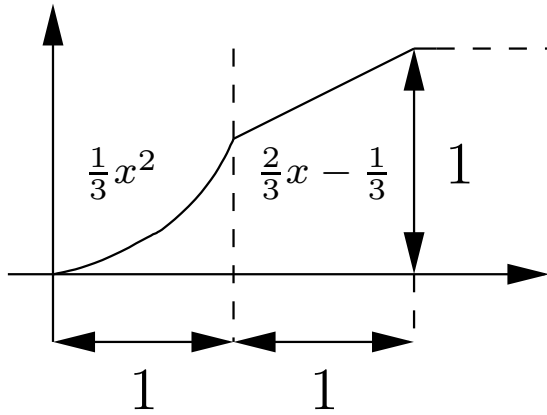
- (a) original image A / B / C / D (circle one)
- (b) image after homomorphic filtering A / B / C / D (circle one)
- (c) result of edge detection A / B / C / D (circle one)
- (d) result of histogram equalization A / B / C / D (circle one)

Question 3



(a) Indicate whether or not each figure above represents a valid probability density function (PDF).

- (A) valid / invalid (circle one) if invalid, state reason why:
- (B) valid / invalid (circle one) if invalid, state reason why:
- (C) valid / invalid (circle one) if invalid, state reason why:
- (D) valid / invalid (circle one) if invalid, state reason why:



(b) Recall that the cumulative distribution function (CDF) of the random variable X is defined as the probability that $X \leq x$. The figure above is the cumulative distribution function corresponding to which of the PDF's in part (a)?

A / B / C / D (circle one)

Question 4

(a) Consider the discrete-time random process, $z[n]$, with

$$\begin{aligned}E\{z[n]\} &= 0 \\E\{z[n]^2\} &= \sigma_z^2 \\E\{z[10]z[11]\} &= (1 - 1/4)\sigma_z^2 \\E\{z[12]z[13]\} &= (1 - 1/4)\sigma_z^2\end{aligned}$$

Which of the following is a correct statement: **(circle one)**

- $z[n]$ is a wide sense stationary process.
- $z[n]$ is not a wide sense stationary process.
- $z[n]$ might be a wide sense stationary process.

(b) Consider the discrete-time random process, $y[n]$. The probability density function (PDF) of $y[n]$ is Gaussian with zero mean and variance, σ^2 , for $n = -\infty, \dots, \infty$. Additionally, the joint PDF of $y[10]$ and $y[11]$ is jointly Gaussian with zero mean and covariance:

$$E\{[y[10] \ y[11]]^T [y[10] \ y[11]]\} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

while the PDF of $y[12]$ and $y[13]$ is jointly Gaussian with zero mean and covariance:

$$E\{[y[12] \ y[13]]^T [y[12] \ y[13]]\} = \begin{bmatrix} \sigma^2 & \frac{\sigma^2}{2} \\ \frac{\sigma^2}{2} & \sigma^2 \end{bmatrix}$$

Which of the following is a correct statement: **(circle one)**

- $y[n]$ is a strict sense stationary process.
- $y[n]$ is not a strict sense stationary process.
- $y[n]$ might be a strict sense stationary process.

(c) Consider the discrete-time random process, $x[n]$, where samples are drawn alternately from either a Gaussian probability density function or a uniform probability density function, so that $x[n]$ is sampled from a Gaussian PDF when n is even and $x[n]$ is sampled from a uniform PDF when n is odd. Both the Gaussian and uniform PDFs have zero mean and variance, σ^2 .

Which of the following are correct statements: **(circle all that apply)**

- $x[n]$ is a strict sense stationary process.
- $x[n]$ is not a strict sense stationary process.
- $x[n]$ might be a strict sense stationary process.
- $x[n]$ is a wide sense stationary process.
- $x[n]$ is not a wide sense stationary process.
- $x[n]$ might be a wide sense stationary process.

Question 5 Problem 1

Figure 1 depicts a system block diagram in which

- $H_1(f)$ and $H_2(f)$ are stable linear shift invariant filters,
- Both $H_1(f) > 0$ and $H_2(f) > 0$ for all f ,
- $x[n]$ is a zero-mean white noise sequence with $\langle x[n]^2 \rangle = 3$,
- $w[n]$ is a zero-mean white noise sequence with $\langle w[n]^2 \rangle = 2$,
- $x[n]$ and $w[n]$ are uncorrelated with each other, and
- $\langle z[n]^2 \rangle = 8$.

Note that you have already worked out some details of this specific diagram from the handout of last year's Quiz 2.

a) Give expressions for the following (in terms of the signals and filters in the diagram):

$$R_{xy}[m] =$$

$$R_y[m] =$$

$$R_{xz}[m] =$$

$$R_z[m] =$$

$$S_{xy}(f) =$$

$$S_{yx}(f) =$$

$$S_{xz}(f) =$$

$$S_y(f) =$$

$$S_z(f) =$$

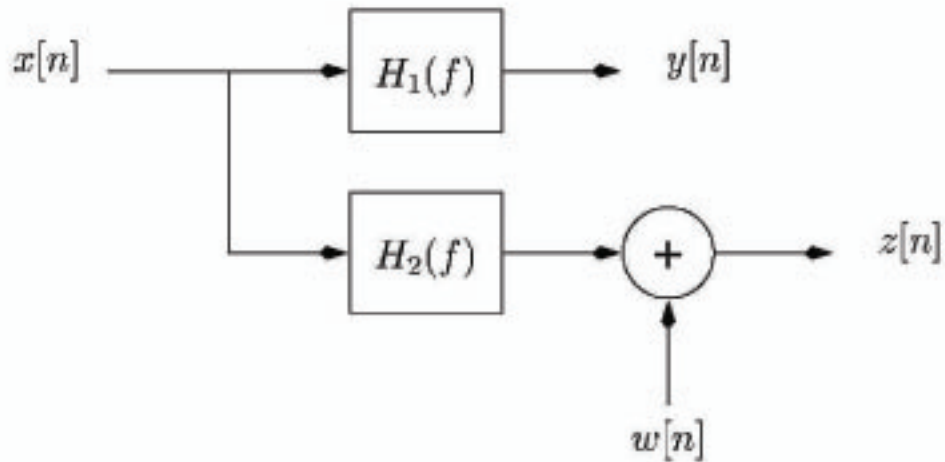


Figure 1: System block diagram for problem 3.

b) Give the frequency domain expression for the wiener filter which estimates:

$$x[n] \text{ from } y[n], H(f) =$$

$$x[n] \text{ from } z[n], H(f) =$$

$$z[n] \text{ from } x[n], H(f) =$$

Question 6

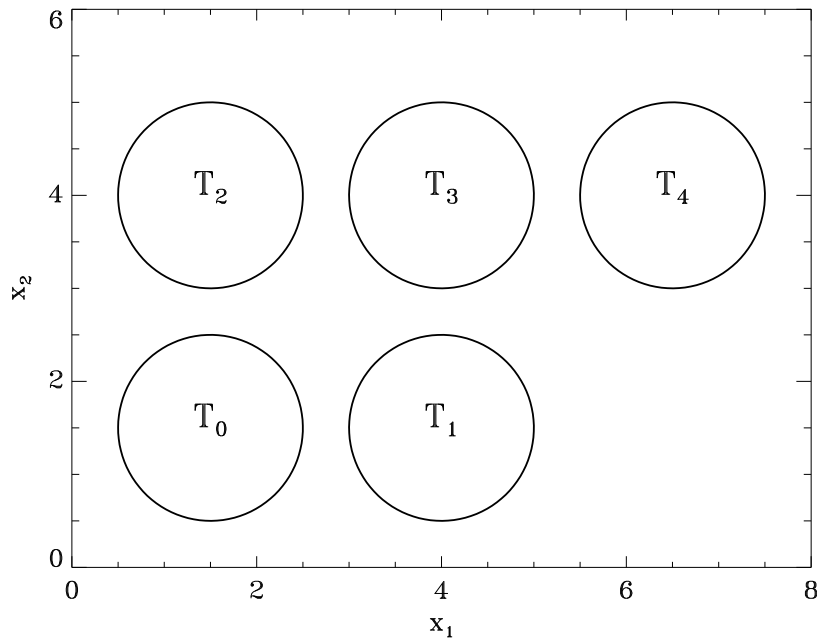


Figure 2: Iso-probability contours

It has been discovered that an individual's blood contains one (and only one) of five antibody types denoted T_0 , T_1 , T_2 , T_3 , and T_4 . We cannot measure the presence of these directly, but we have two blood tests, denoted X_1 and X_2 , which yield numerical values, x_1 and x_2 , according to the conditional probability density functions:

$$p_{X_1, X_2}(x_1, x_2 | T = T_i) = N \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; M_i, \Sigma_i \right)$$

where $N(\mathbf{x}; M_i, \Sigma_i)$ is a 2-dimensional Gaussian density function with mean, M_i , and covariance, Σ_i . The class-conditional means and covariances, denoted by their index are:

$$\Sigma_0 = \Sigma_1 = \Sigma_2 = \Sigma_3 = \Sigma_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M_0 = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix} \quad M_1 = \begin{bmatrix} 4 \\ 1.5 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1.5 \\ 4 \end{bmatrix} \quad M_3 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad M_4 = \begin{bmatrix} 6.5 \\ 4 \end{bmatrix}$$

Iso-probability contours for each of the conditional densities are shown in Figure 4 for your reference. This figure will be repeated for each part of this question in the event that you wish to mark it.

In the problems that follow we will consider subgroups (i.e. classes) of the population at large which contain individuals with different antibody types. **In all cases, the prior probability of each class is 0.5.**

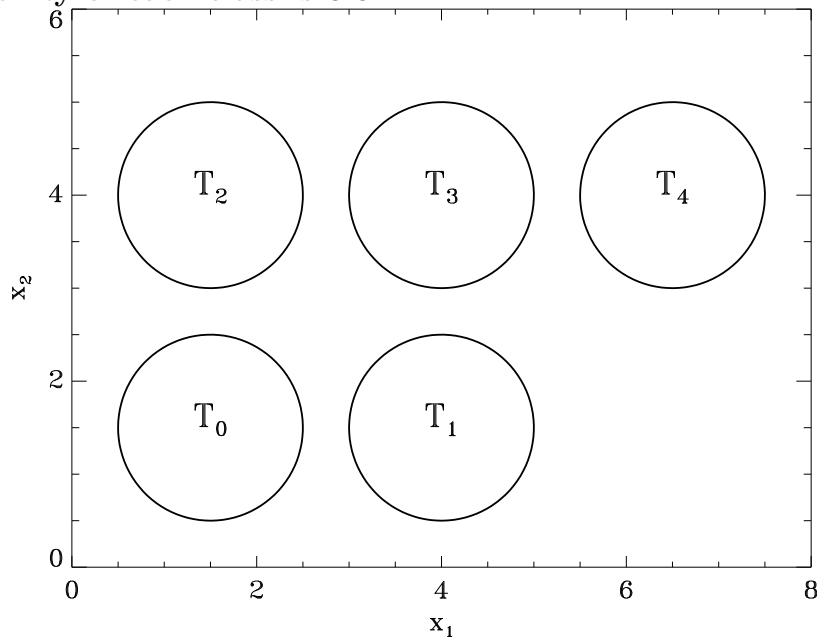


Figure 3: Iso-probability contours

(a) **AB-Classifier:** Consider the population where each individual has either the T_0 , T_1 , T_2 , or T_3 antibody, with equal probability. In other words, 25% of the population has T_0 , 25% has T_1 , 25% has T_2 , 25% has T_3 , and no one has T_4 . We divide this population into two classes. Class **A** individuals have either the T_0 antibody or the T_1 antibody, while class **B** individuals have either the T_2 antibody or the T_3 antibody. Which tests are *necessary* to achieve a minimum classification error rate when distinguishing class **A** from class **B**? **(circle one)**

- X_1 only
- X_2 only
- both X_1 and X_2
- neither test is necessary

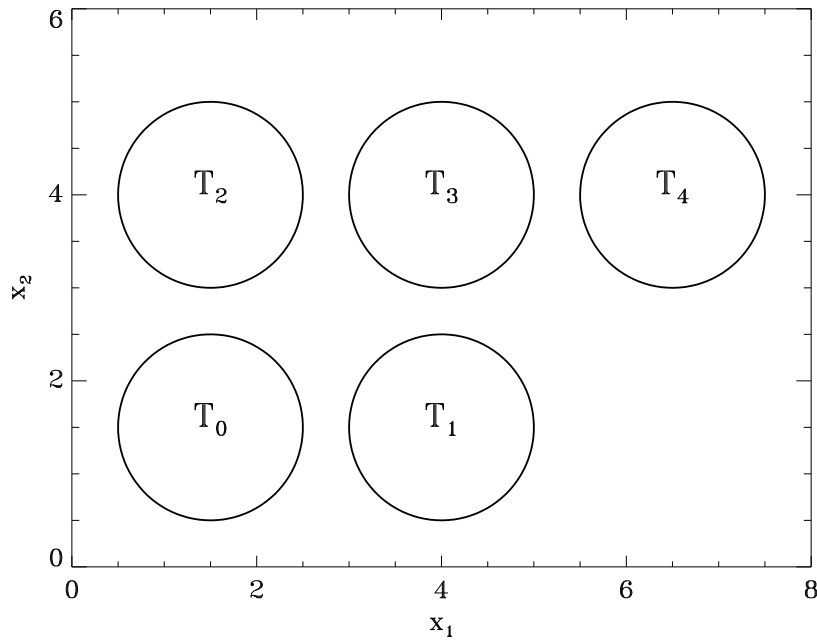


Figure 4: Iso-probability contours

(b) CD-Classifier: Again, consider the population where each individual has either the T_0 , T_1 , T_2 , or T_3 antibody, with equal probability. In other words, 25% of the population has T_0 , 25% has T_1 , 25% has T_2 , 25% has T_3 , and no one has T_4 . We divide this population into two classes. Class **C** individuals have either the T_1 antibody or the T_2 antibody, while class **D** individuals have either the T_0 antibody or the T_3 antibody. Which tests are *necessary* to achieve a minimum classification error rate when distinguishing class **C** from class **D**? (**circle one**)

- X_1 only
- X_2 only
- both X_1 and X_2
- neither test is necessary

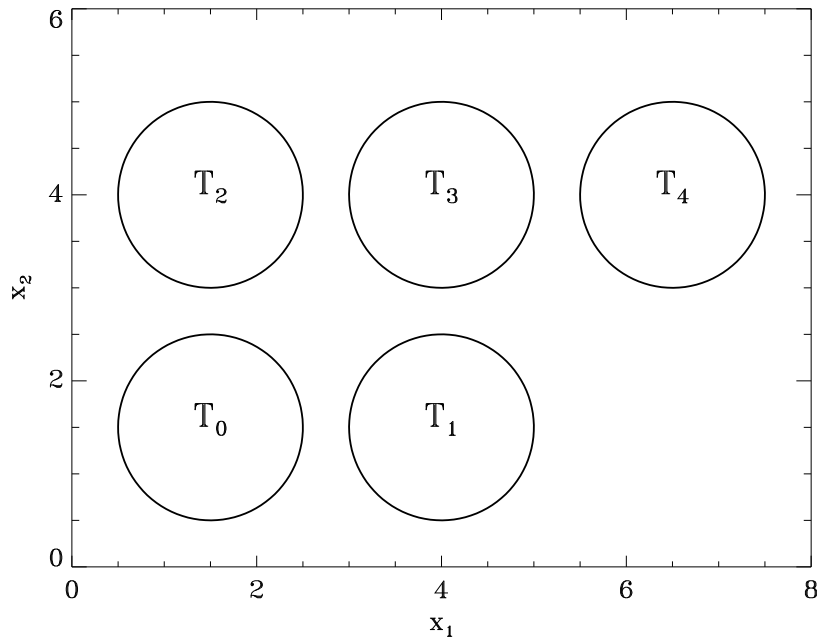


Figure 5: Iso-probability contours

(c) **AE-Classifier part 1:** Now consider the population where each individual may have any of the antibodies, T_0 through T_4 , but with unequal probabilities: 25% of the population has T_0 , 25% has T_1 , 20% has T_2 , 20% has T_3 , and 10% has T_4 . We divide this population into two classes. Class **A** individuals have either the T_0 or T_1 antibody, as before, while class **E** individuals have either the T_2 , T_3 , or T_4 antibody. (Note that the prior probabilities of class **A** and class **E** are each 0.5.) We use the classification rule for the **AB-classifier** (from part (a) of this problem) to classify individuals as class **A** or class **E**, so that those individuals which would be labeled as class **B** will now be labeled as class **E**). This classifier will result in a classification error rate which is: **(circle one)**

- higher than the **AB**-classification error rate.
- lower than the **AB**-classification error rate.
- the same as the **AB**-classification error rate.

(d) **AE-Classifier part 2:** Using the classes **A** and **E** as defined in part (c), the minimum achievable classification error rate for classes **A** or class **E** as defined above is: **(circle one)**

- higher than the **AB**-classification error rate.
- lower than the **AB**-classification error rate.
- the same as the **AB**-classification error rate.

NAME:

HST-582J/6.555J/16.456J

Biomedical Signal and Image Processing

Spring 2006

Quiz 2 — May 11, 2006

- There are **4** questions on **9** pages.
- Please write all of your answers and explanations directly on this quiz form. **NO** credit will be given for anything written on scratch paper.
- Explanations should be brief and to the point. Unnecessarily long or wordy justifications may be penalized.

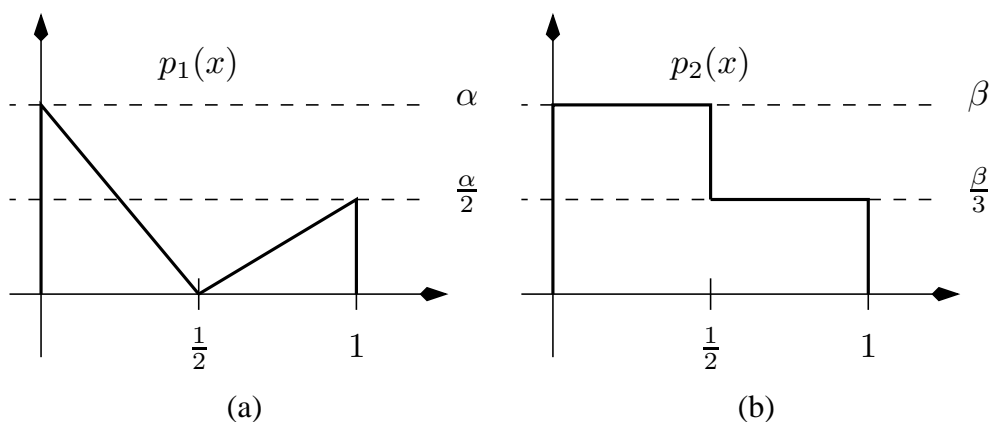


Figure 1: PDFs for Question 1. Note that figures are not necessarily to scale.

Question 1 (20 %)

Figures 1(a) and 1(b) depict probability density functions. **(Your answers to parts (b)-(d) must be consistent with your answer to part(a))**

- (a) Determine constants α and β such that $p_1(x)$ and $p_2(x)$ are valid probability density functions.

$$\alpha = \underline{\hspace{2cm}} \quad \beta = \underline{\hspace{2cm}}$$

- (b) The PDF of the random variable z is:

$$p(z) = \frac{1}{2}p_1(z) + \frac{1}{2}p_2(z)$$

Compute the expected value of z .

$$E\{z\} = \underline{\hspace{2cm}}$$

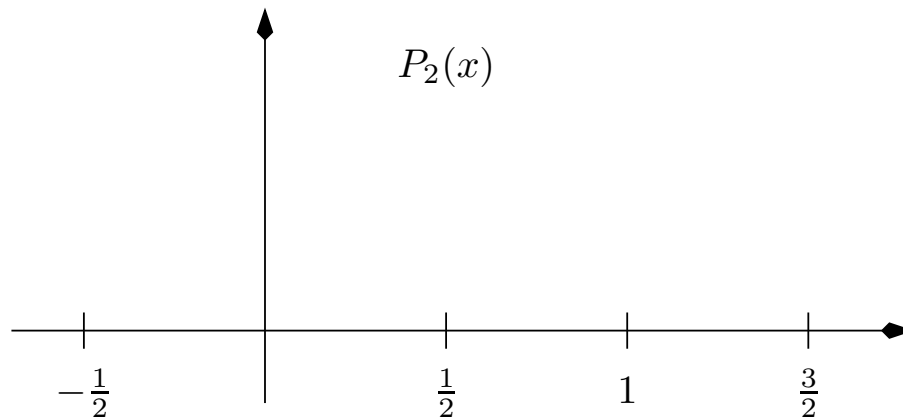
- (c) Let $y = \frac{1}{2}x_1 + \frac{1}{2}x_2$, where

$$x_1 \sim p_1(x) \quad x_2 \sim p_2(x) \quad \text{and} \quad p(x_1, x_2) = p_1(x_1)p_2(x_2)$$

Compute the expected value of Y .

$$E\{y\} = \underline{\hspace{2cm}}$$

(d) Sketch the **cumulative** distribution function corresponding to the probability density function, $p_2(x)$, over the range $-\frac{1}{2} < x < \frac{3}{2}$. Clearly, label important details in your plot.



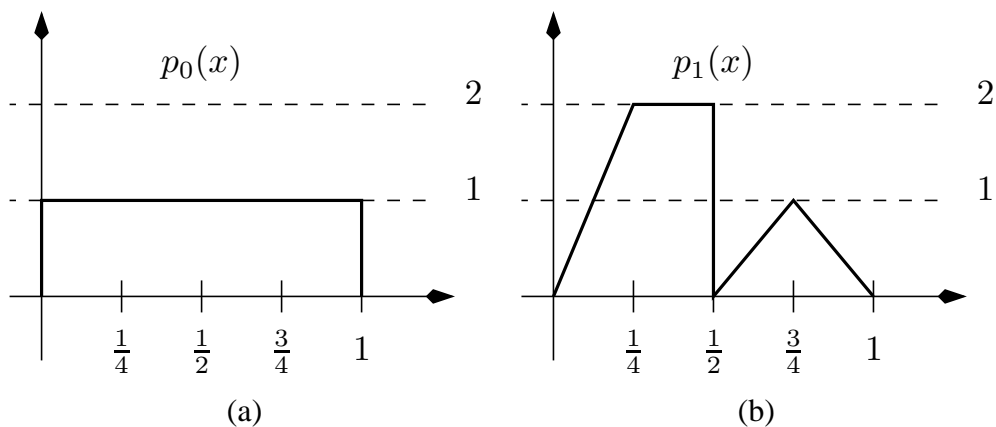


Figure 2: PDFs for Question 2. Note that figures are not necessarily to scale.

Question 2 (30 %)

Figures 2(a) and 2(b) depict **class-conditional** probability density functions.

This question concerns the binary hypothesis test where:

$$H_0 : x \sim p_0(x)$$

$$H_1 : x \sim p_1(x)$$

The prior probabilities of hypotheses H_0 and H_1 are denoted P_0 and P_1 respectively.

For parts (c)-(e) define:

$$P_{FA} = \Pr \{ \text{choosing } H_1 | H_0 \text{ is true} \}$$

$$P_D = \Pr \{ \text{choosing } H_1 | H_1 \text{ is true} \}$$

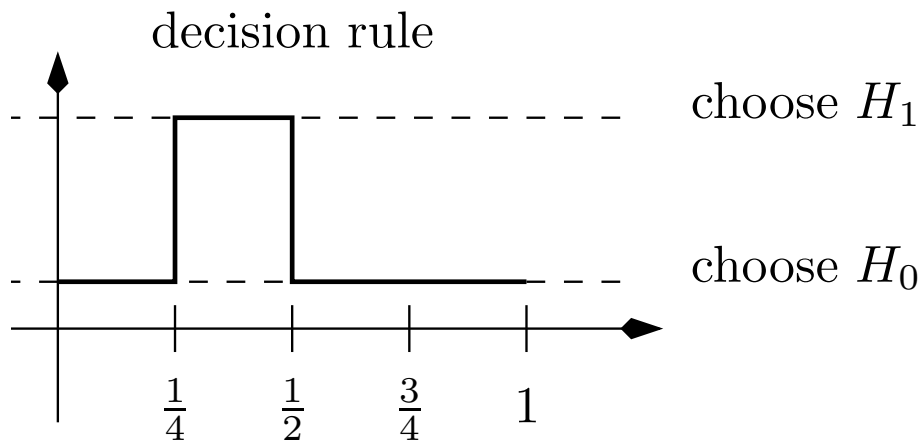
- (a) Determine the minimum value of P_0 such that the decision rule would be to always assign a measurement x to H_0 .

$$\min P_0 = \underline{\hspace{2cm}}$$

- (b) Determine the minimum value of P_1 such that the decision rule would be to always assign a measurement x to H_1 .

$$\min P_1 = \underline{\hspace{2cm}}$$

- (c) Given $P_1 = \frac{1}{2}$, a 1-nearest neighbor classifier rule based on a set of labeled samples results in the following decision rule:



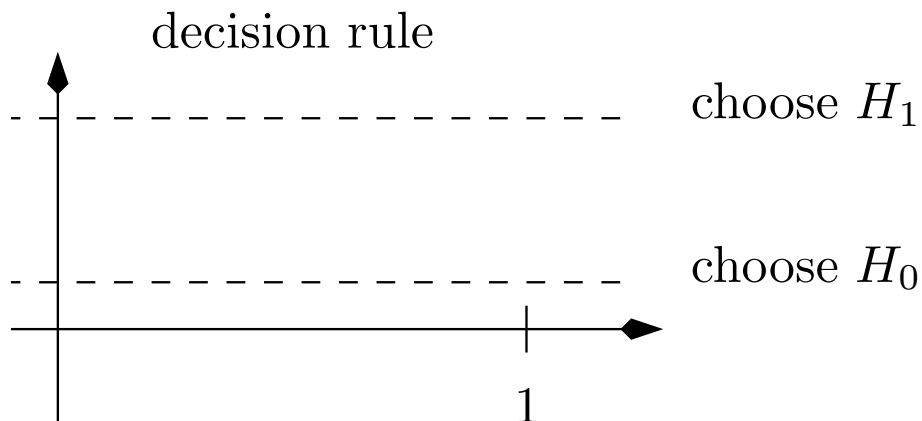
For this decision rule compute:

$$P_{FA} = \underline{\hspace{2cm}} \quad P_D = \underline{\hspace{2cm}} \quad P_E = \underline{\hspace{2cm}}$$

- (d) Consider the decision rule that is the reverse of part (c). That is, it chooses H_0 when the decision rule of part (c) chooses H_1 and chooses H_1 when the decision rule of part (c) chooses H_0 , compute:

$$P_{FA} = \underline{\hspace{2cm}} \quad P_D = \underline{\hspace{2cm}} \quad P_E = \underline{\hspace{2cm}}$$

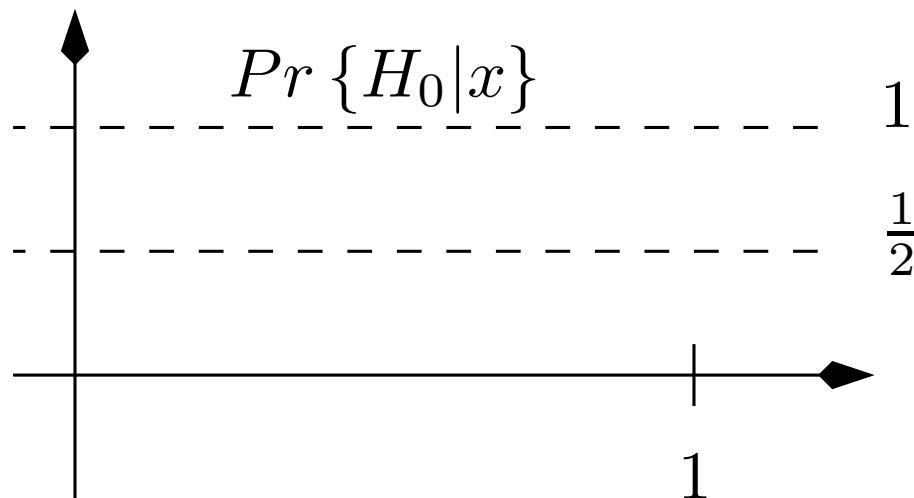
(e) On the plot provided, sketch the **optimal** decision rule. Assume $P_1 = \frac{1}{2}$. Be sure to clearly label all points.



For the optimal decision rule compute:

$P_{FA} = \underline{\hspace{2cm}}$ $P_D = \underline{\hspace{2cm}}$ $P_E = \underline{\hspace{2cm}}$

(f) On the plot provided, sketch the posterior probability of H_0 as a function of x . Assume $P_1 = \frac{1}{2}$. Be sure to clearly label all points.



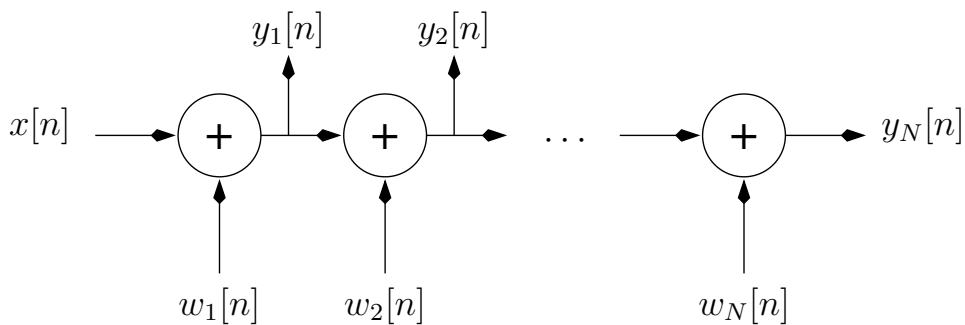


Figure 3: System Block Diagram for Question 3.

Question 3 (20 %)

Figure 3 depicts a system block diagram for the set of random processes for which:

$$y_i[n] = x[n] + \sum_{k=1}^i w_k[n]$$

where

- $x[n]$ is a zero-mean WSS random process with autocorrelation function $R_x[m]$.
- $w_i[n]$'s are zero-mean white noise sequences with $\langle w_i[n]^2 \rangle = \frac{1}{2}$.
- $w_i[n]$'s are uncorrelated with each other and with $x[n]$.

Using the information given, find expressions for the following:

$$R_{xy_2}[m] = \underline{\hspace{2cm}}$$

$$R_{xy_N}[m] = \underline{\hspace{2cm}}$$

$$R_{y_2}[m] = \underline{\hspace{2cm}}$$

$$R_{y_N}[m] = \underline{\hspace{2cm}}$$

$$\langle (y_{10}[n] - y_5[n])^2 \rangle = \underline{\hspace{2cm}}$$

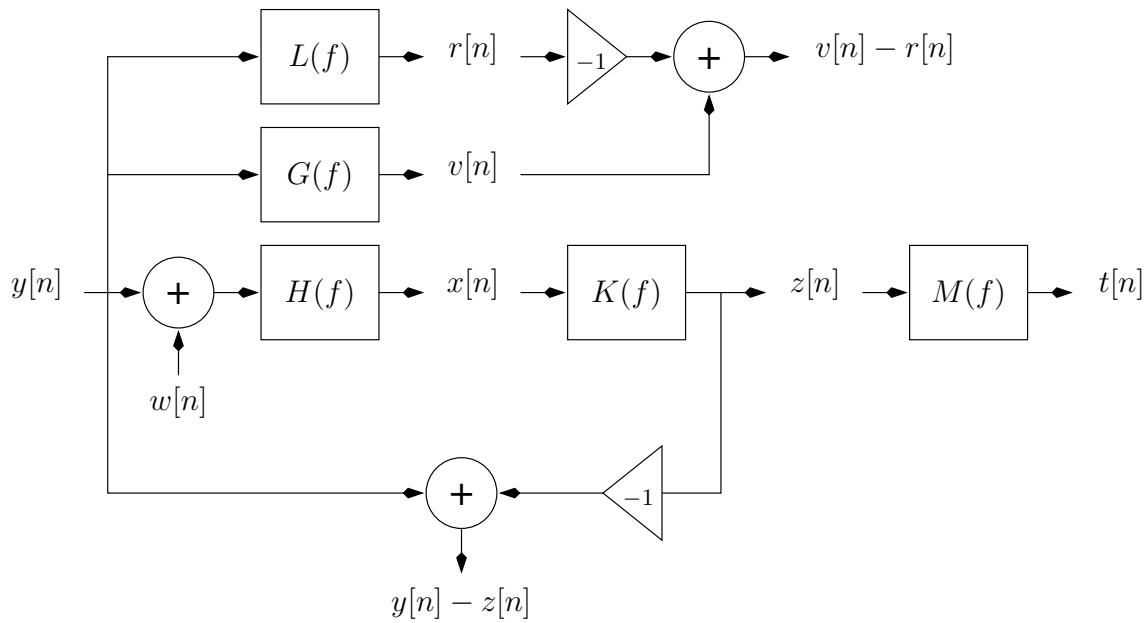


Figure 4: System Block Diagram for Question 4.

Question 4 (30 %)

All parts of Question 4 refer to the system block diagram of Figure 4.

- $H(f), G(f)$ are stable linear shift-invariant filters.
- $H(f), G(f)$ are real-valued with $H(f), G(f) > 0$ for all f .
- $H(f) = H(-f), G(f) = G(-f)$.
- $y[n]$ is a zero-mean WSS random process with autocorrelation function $R_y[m]$.
- $w[n]$ is a zero-mean white noise sequence with $\langle w[n]^2 \rangle = 2$.
- $y[n]$ and $w[n]$ are uncorrelated with each other.

(a) Using the information given, determine $L(f)$ such that $\langle (r[n] - v[n])^2 \rangle$ is minimized.

$$L(f) = \underline{\hspace{10em}}$$

(b) Using the information given, determine $K(f)$ such that $\langle (y[n] - z[n])^2 \rangle$ is minimized.

$$K(f) = \underline{\hspace{10em}}$$

(c) Given the optimal $K(f)$ from part (b), suppose you wish to design $M(f)$ to minimize $\langle (v[n] - t[n])^2 \rangle$. Will $M(f)$ differ from $L(f)$ designed in part (a)?

(circle one) YES / NO

Explain your answer: